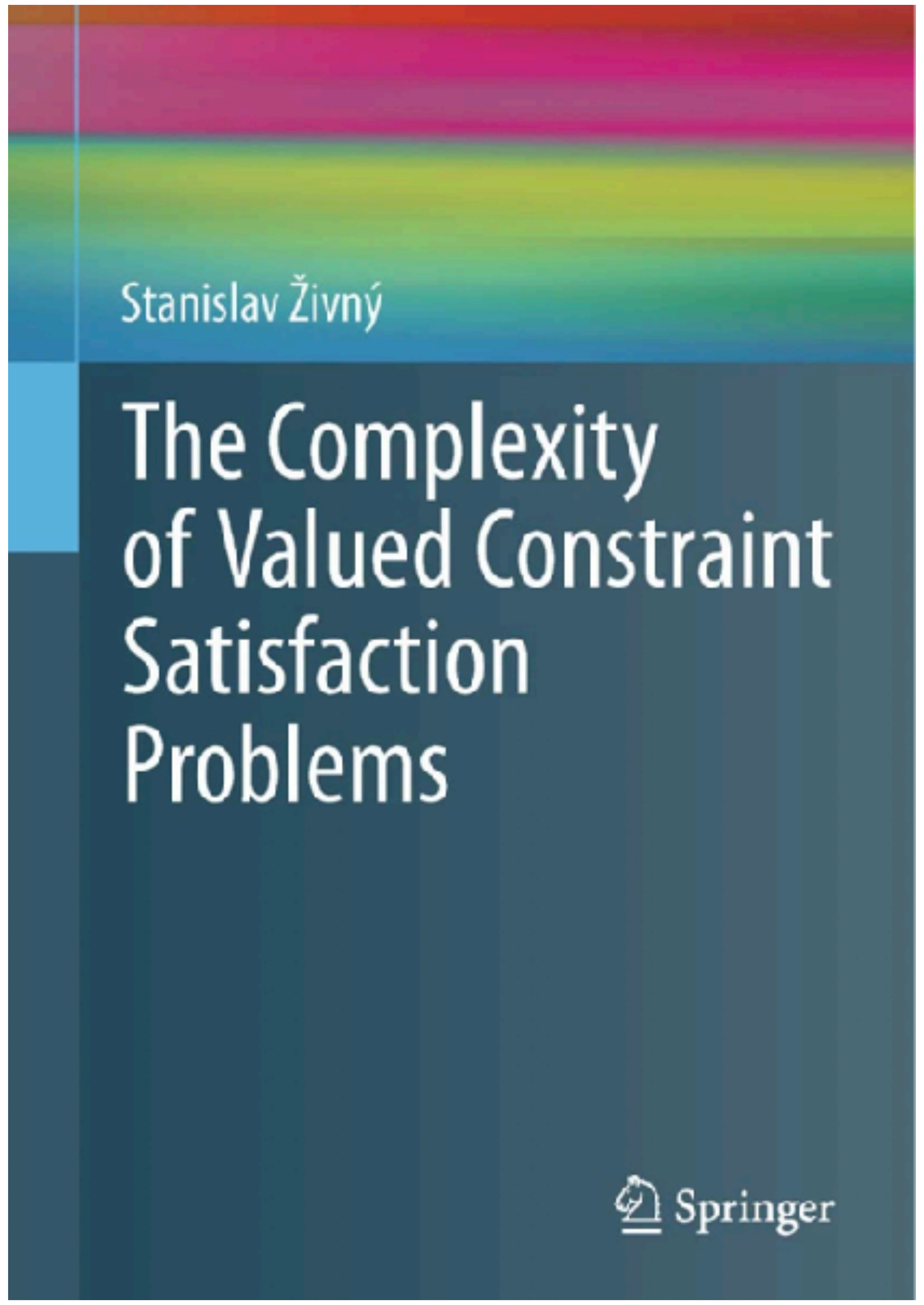
Promise Constraint Satisfaction Problems

Standa Živný (Oxford)

Liblice, 27th September 2023

January 2009

January 2009



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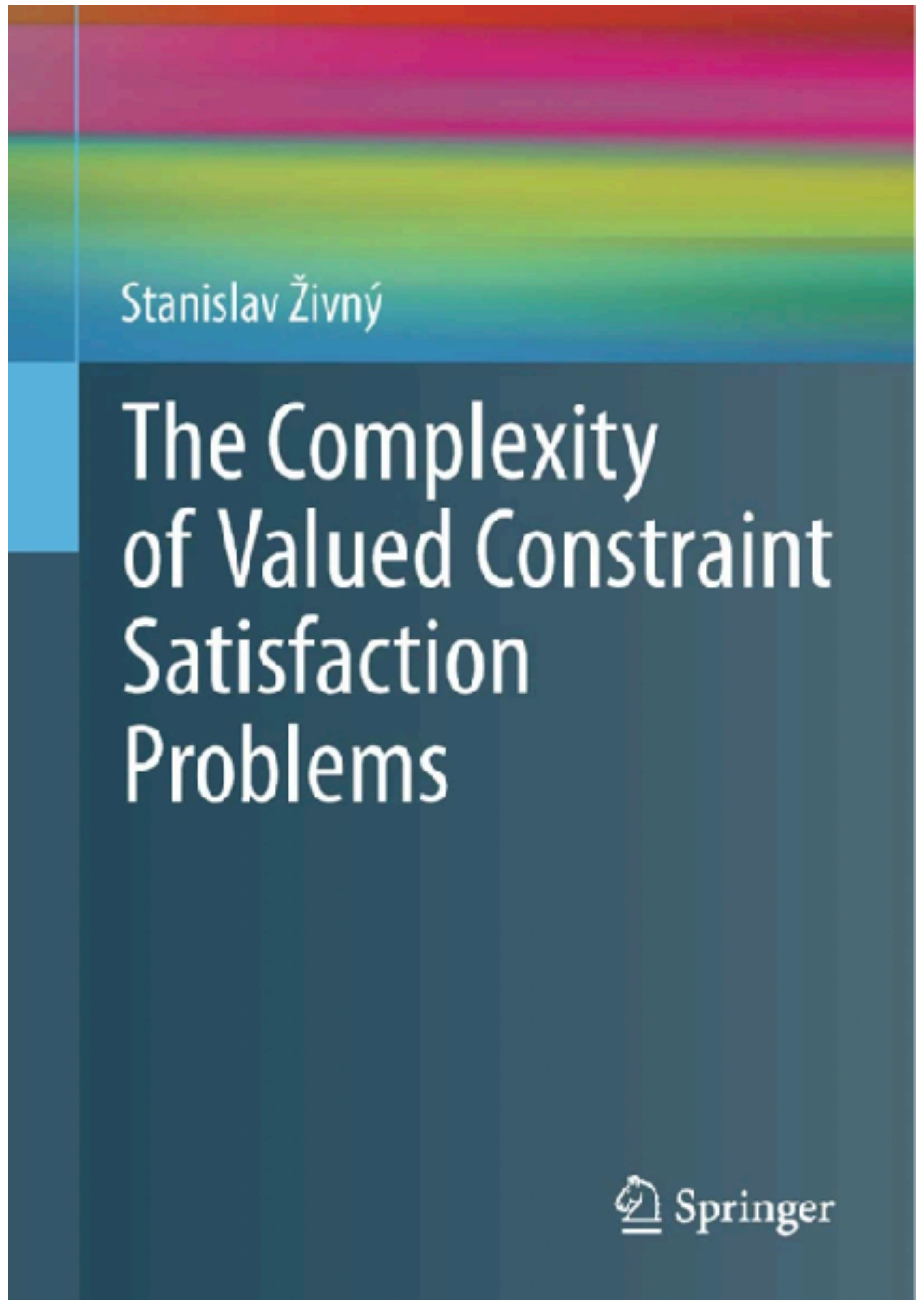

January 2009

DIMACS

Center for Discrete Mathematics & Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center

DIMACS-RUTCOR Workshop on Boolean and Pseudo-Boolean Functions in Memory of Peter L. Hammer

January 19 - 22, 2009
University Inn and Conference Center
New Brunswick Campus, Rutgers University

Organizers:

Endre Boros, Director of RUTCOR, Endre.Boros at rutcor.rutgers.edu

January 2009

ELSEVIER

DISCRETE APPLIED MATHEMATICS

Discrete Applied Mathematics 123 (2002) 155–225

Pseudo-Boolean optimization ☆

Endre Boros*, Peter L. Hammer

RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA

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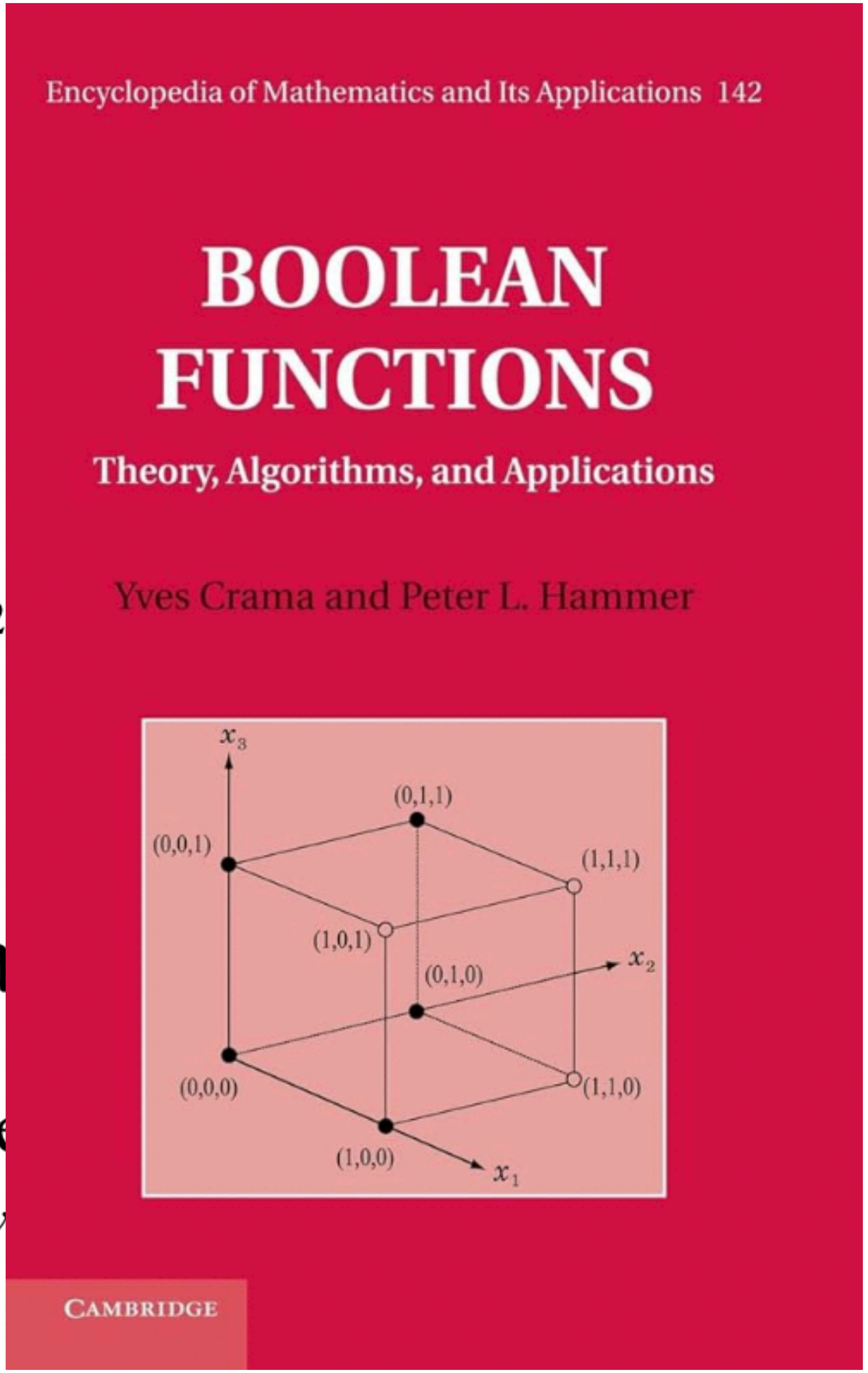

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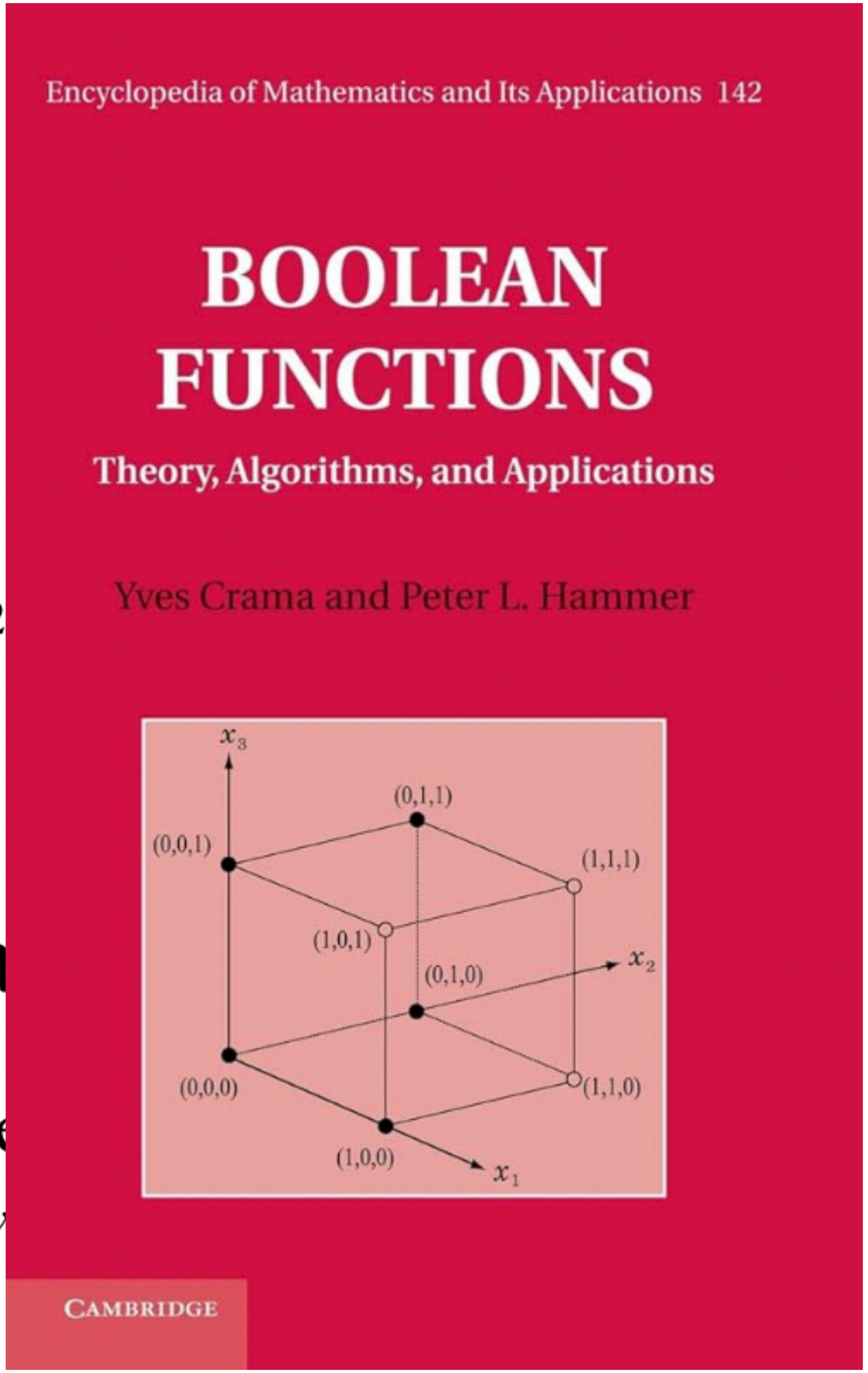

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1. Examples

7-SAT

$$(x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4 \vee \bar{x}_5 \vee x_6 \vee x_7)$$

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Thm: (1,g,k)-SAT in P if $\frac{g}{k} \geq \frac{1}{2}$ and NP-hard otherwise. *[Austrin-Guruswami-Håstad SICOMP'17]*

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NP-hardness of (1,g,2g+1)-SAT.

(a,g,k) $\equiv_p$ (a+1,g+1,k+1)

7-SAT

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Thm: (1,g,k)-SAT in P if $\frac{g}{k} \geq \frac{1}{2}$ and NP-hard otherwise. *[Austrin-Guruswami-Håstad SICOMP'17]*

NP-hardness of (1,g,2g+1)-SAT.

1-in-3-SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5)$$

NAE-3-SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5)$$

1-in-3/NAE-3-SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5)$$

1-in-3/NAE-3-SAT

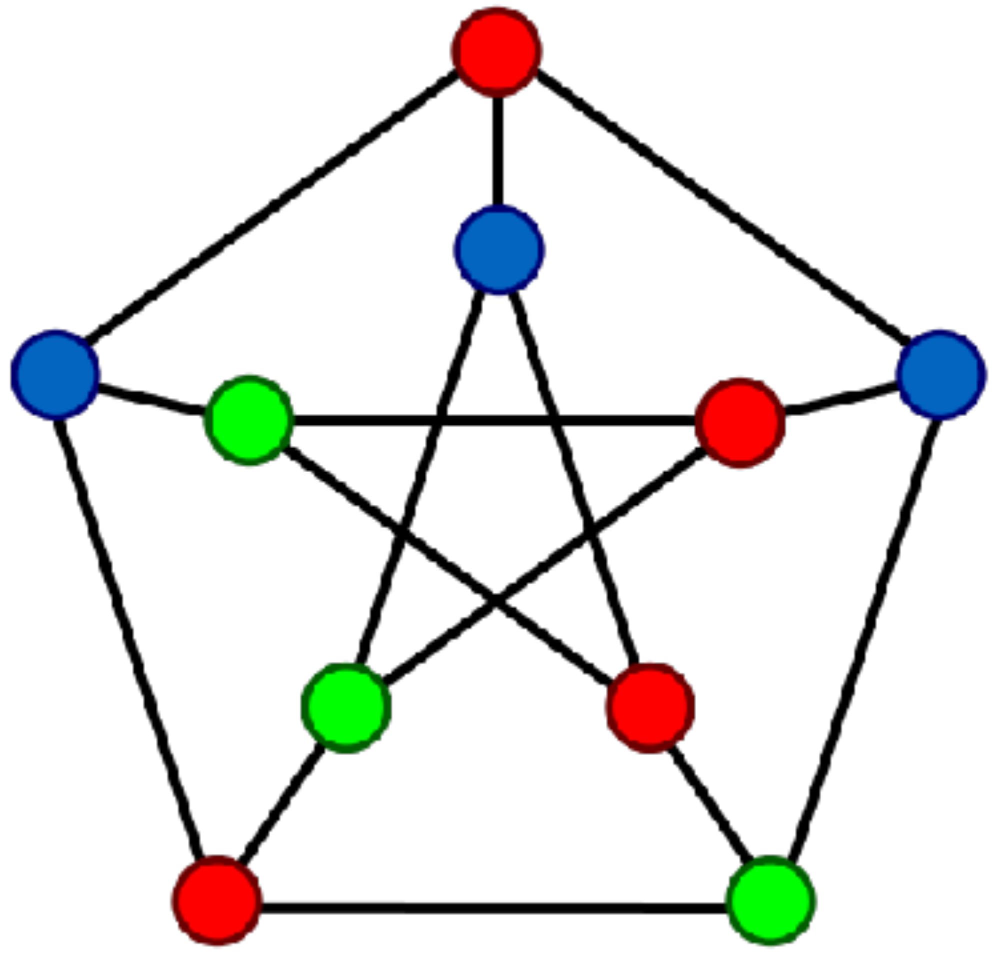
$$(x_1 \lor x_2 \lor x_3) \land (x_2 \lor x_4 \lor x_5)$$

1-in-3/NAE-3-SAT

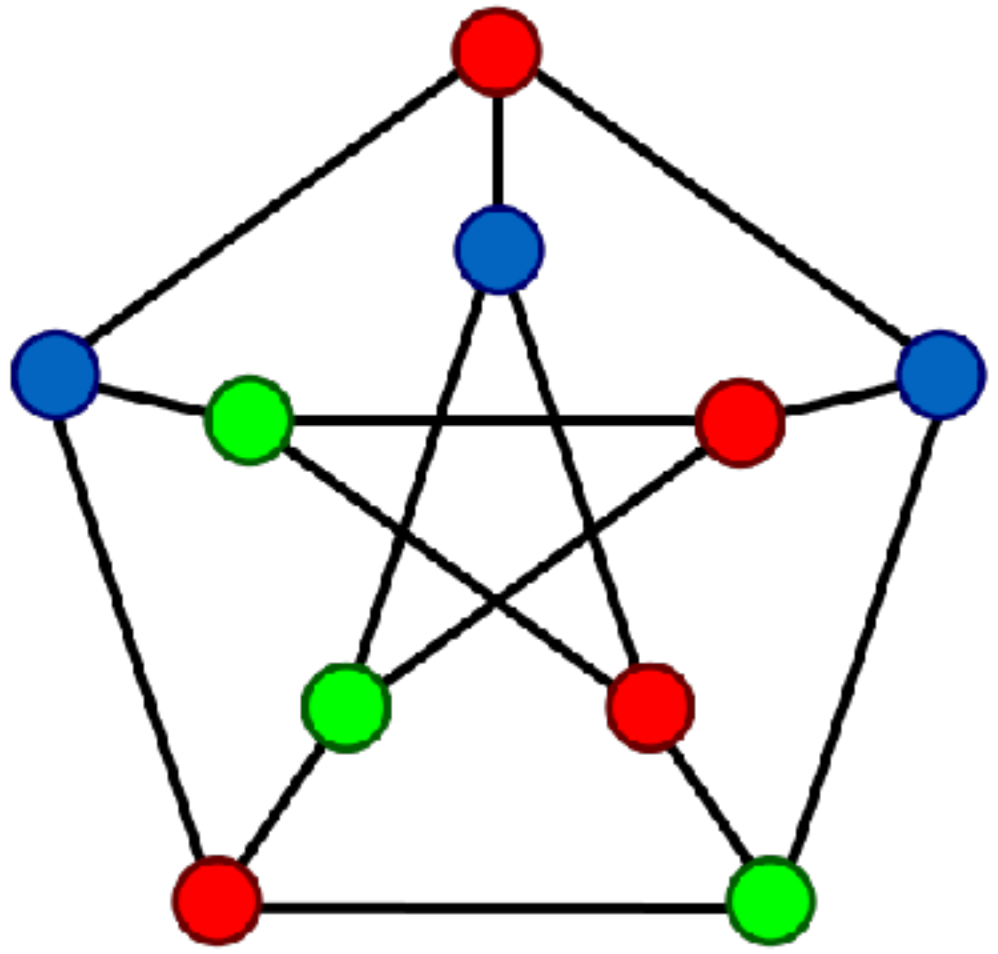
$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5)$$

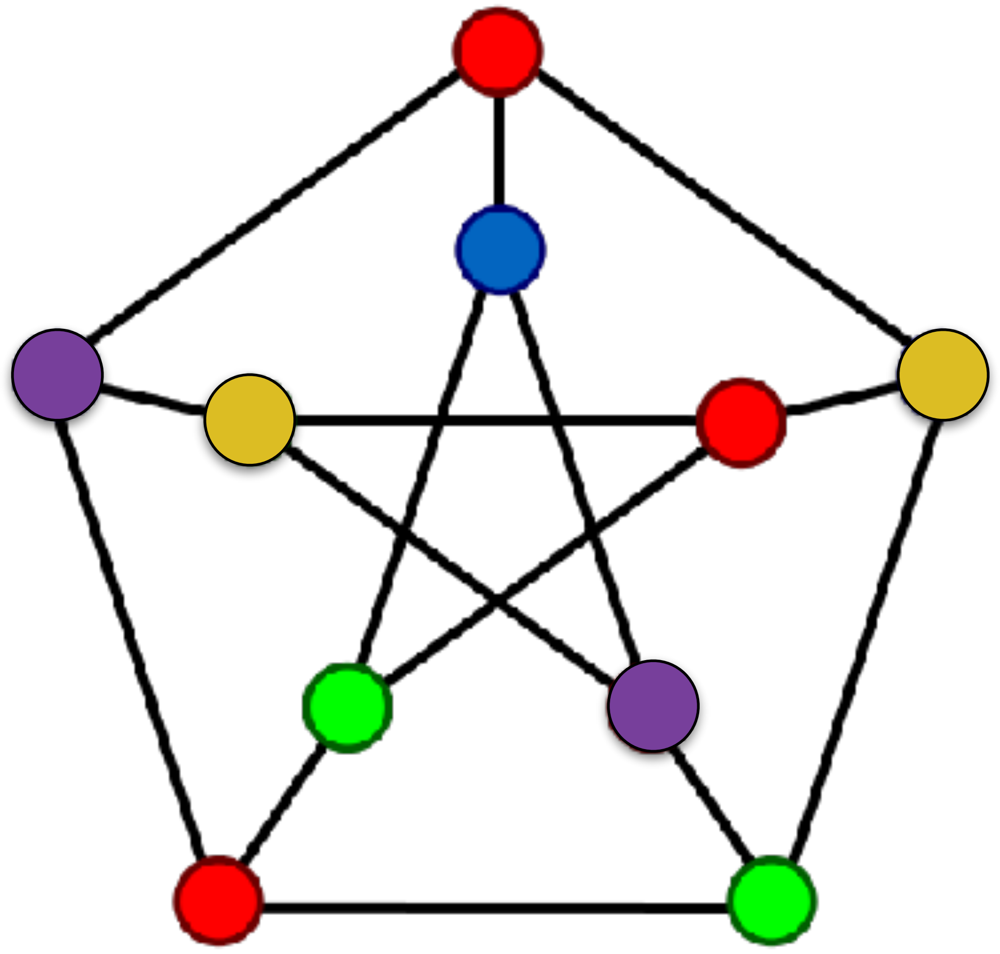
Thm: 1-in-3/NAE-SAT is in P. *[Brakensiek-Guruswami SICOMP'21]*

Graph Colouring

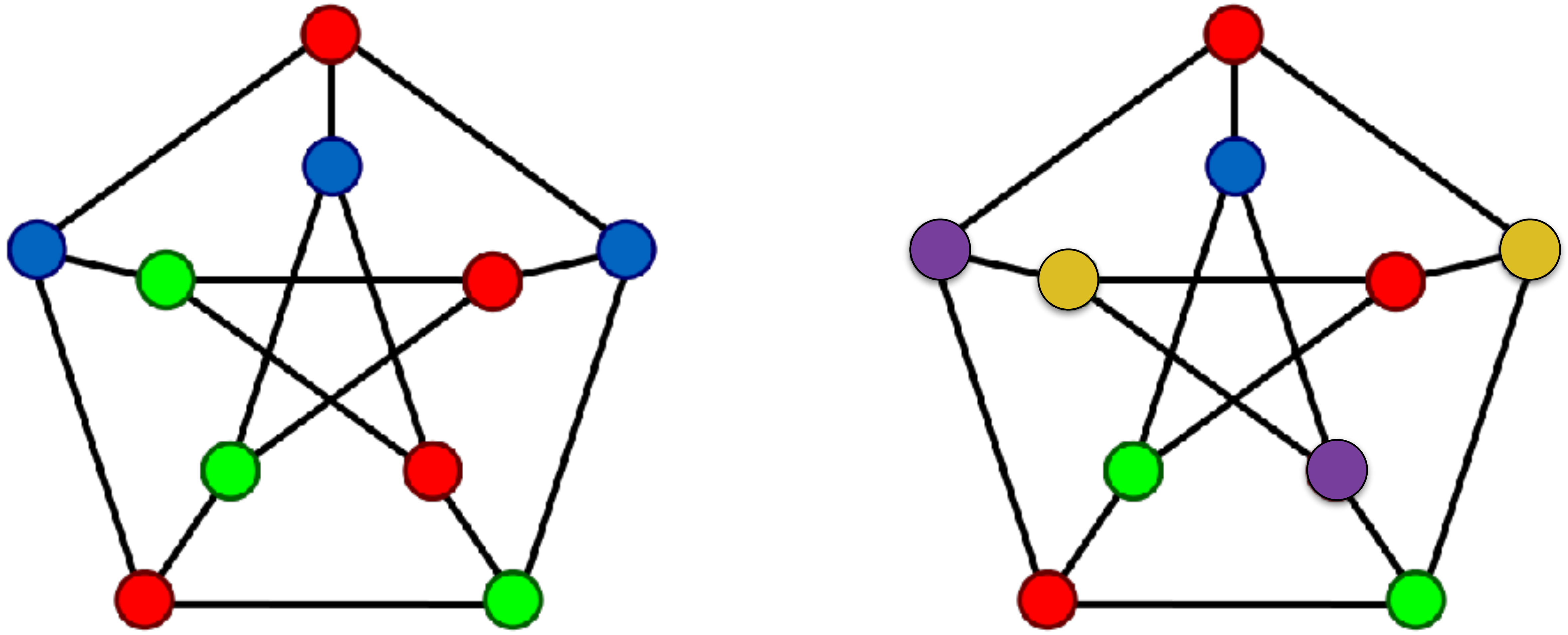


Graph Colouring



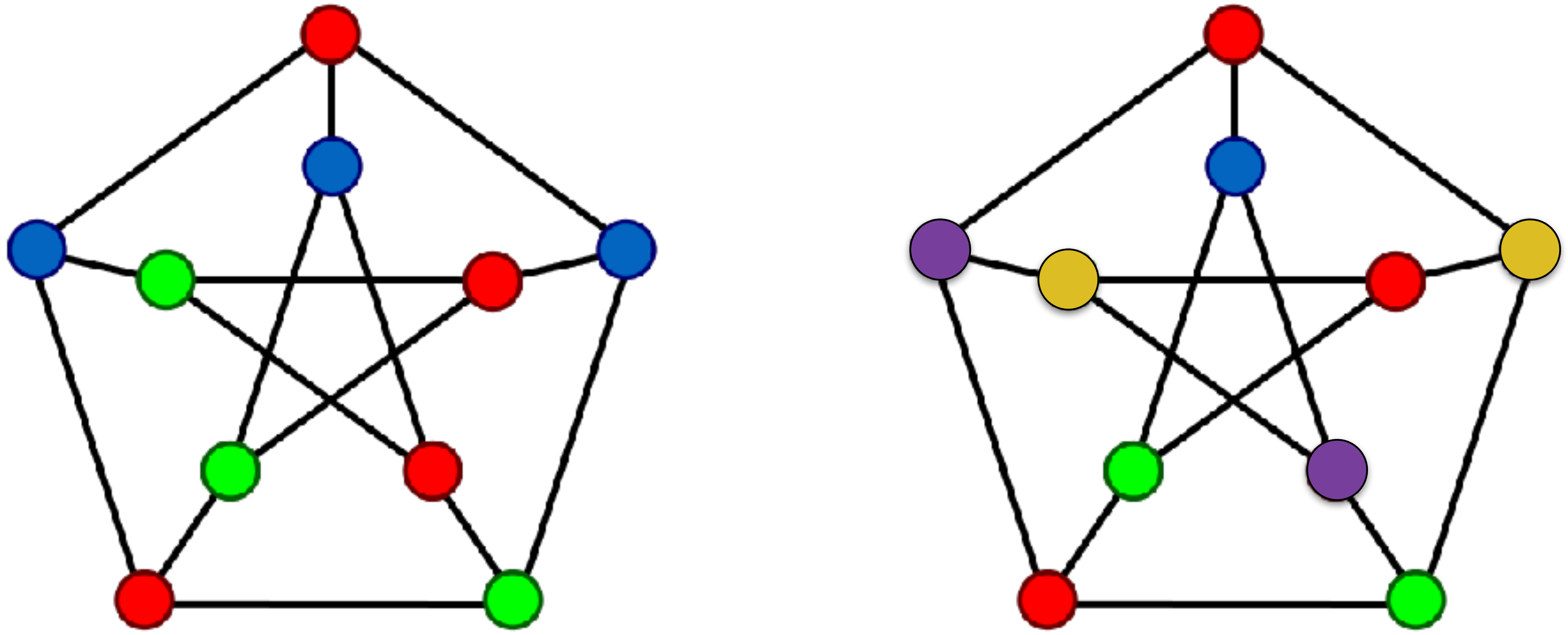


Graph Colouring



Find a c-colouring of a k-colourable graph.

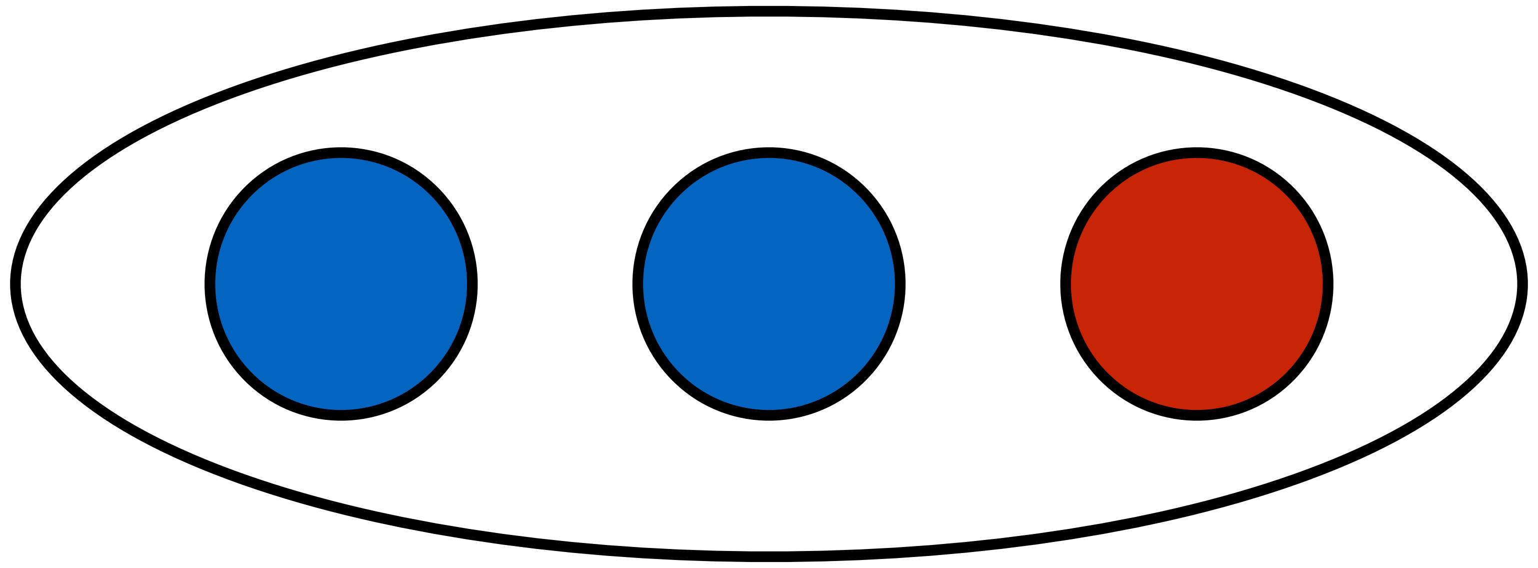
Graph Colouring



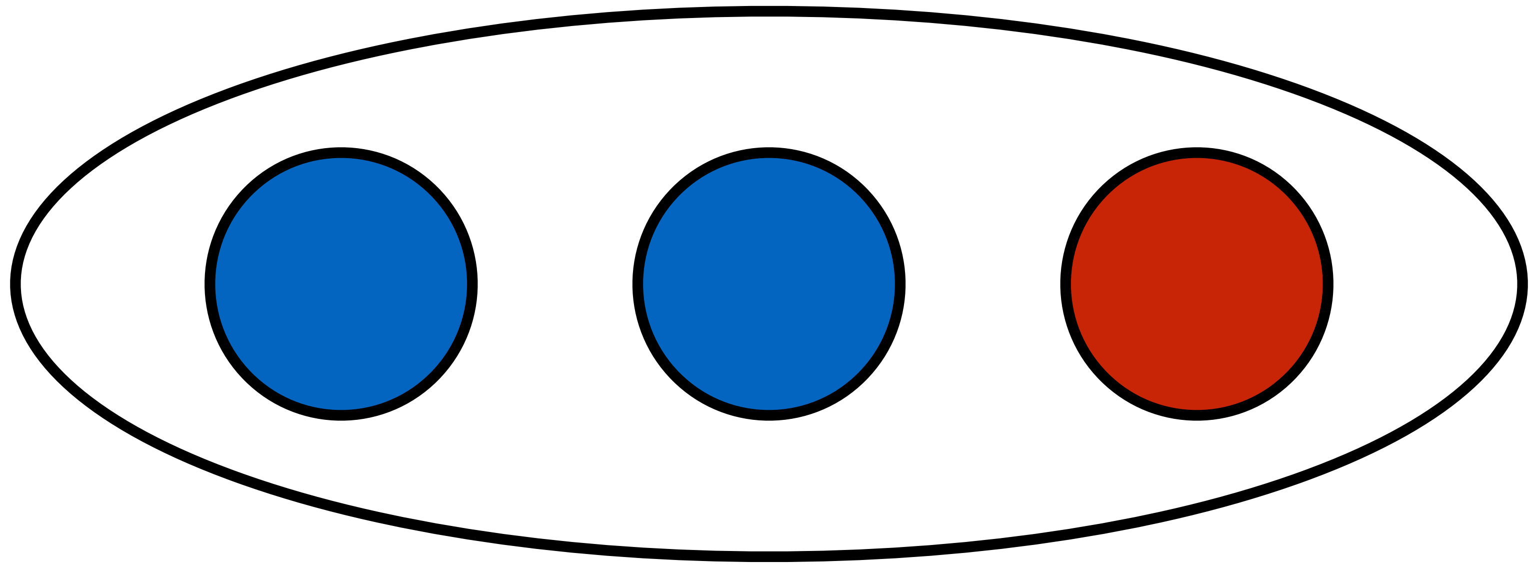
Find a c-colouring of a k-colourable graph.

Conjecture: NP-hard for every constant $c \geq k \geq 3$. *[Garey-Johnson JACM'76]*

Hypergraph Colouring

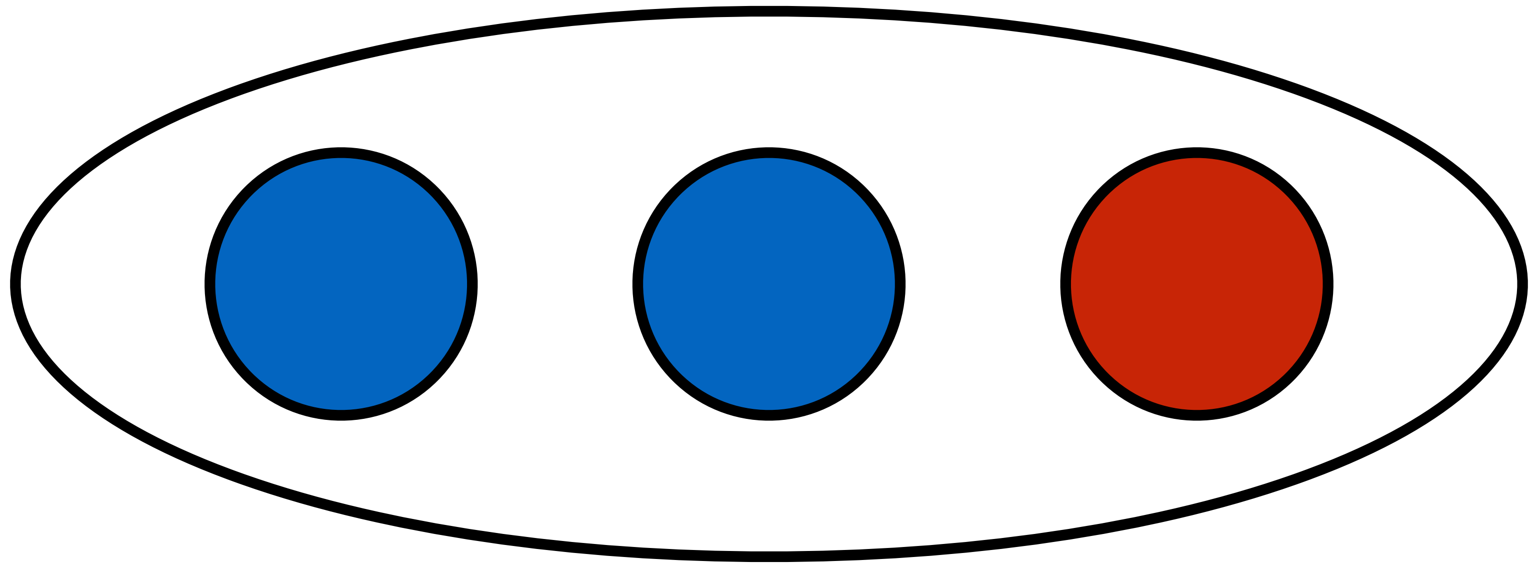


Hypergraph Colouring



Find a c-colouring of a k-colourable 3-uniform hypergraph.

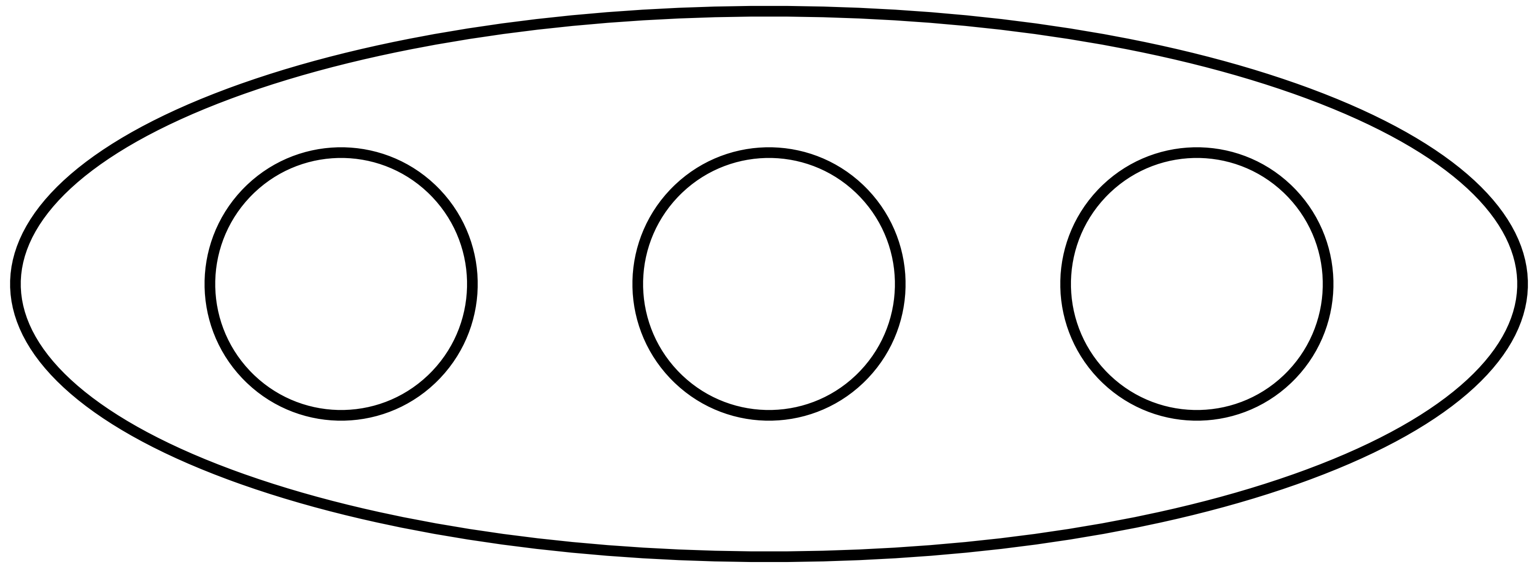
Hypergraph Colouring



Find a c-colouring of a k-colourable 3-uniform hypergraph.

Thm: NP-hard for every constant $c \geq k \geq 2$. *[Dinur-Regev-Smyth Combinatorica'05]*

LO Colouring



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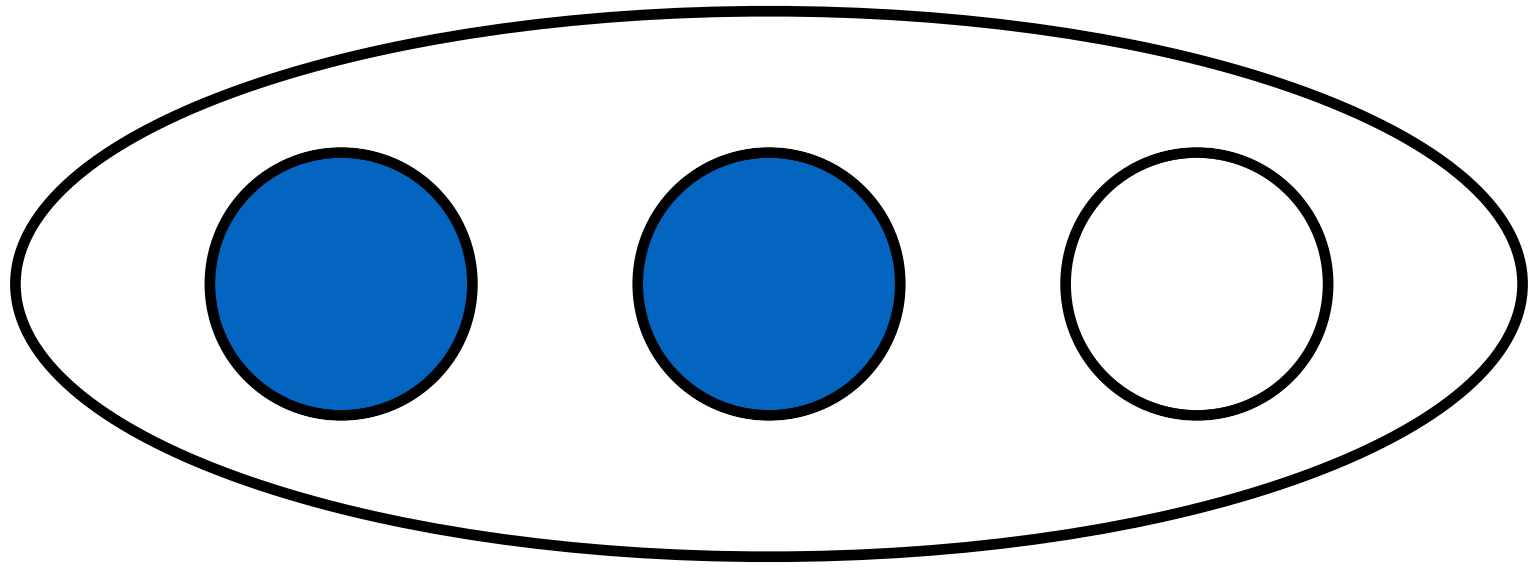

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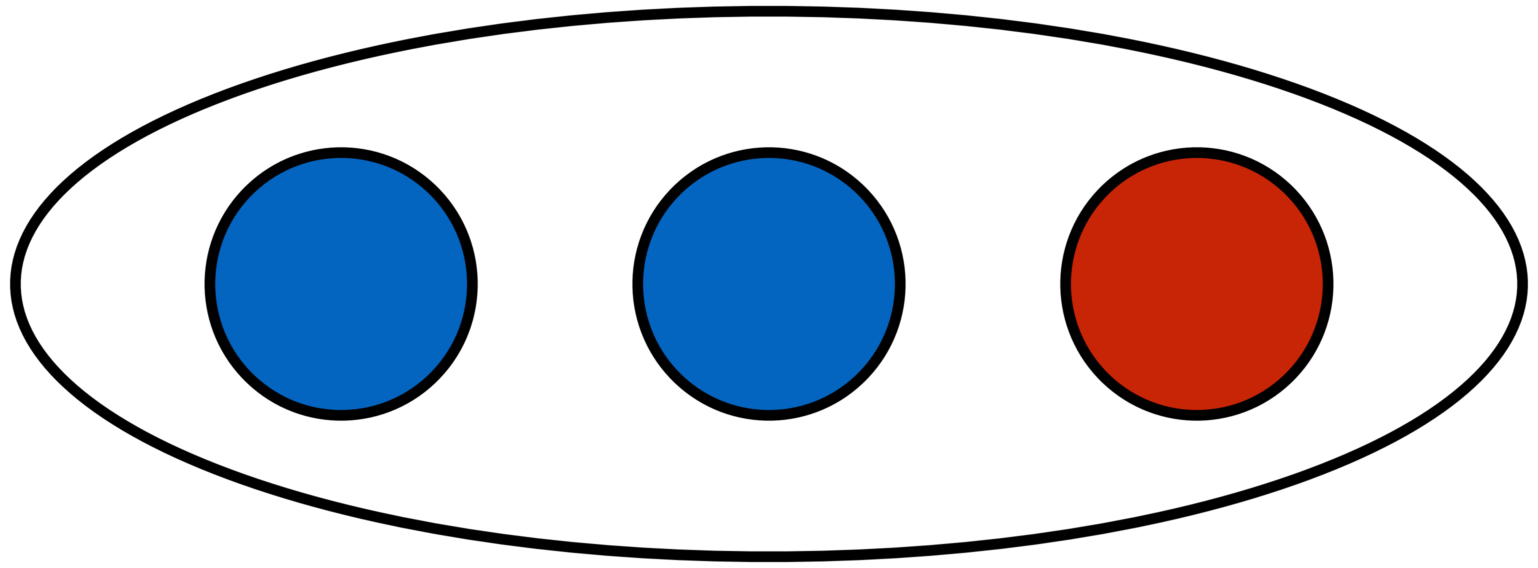

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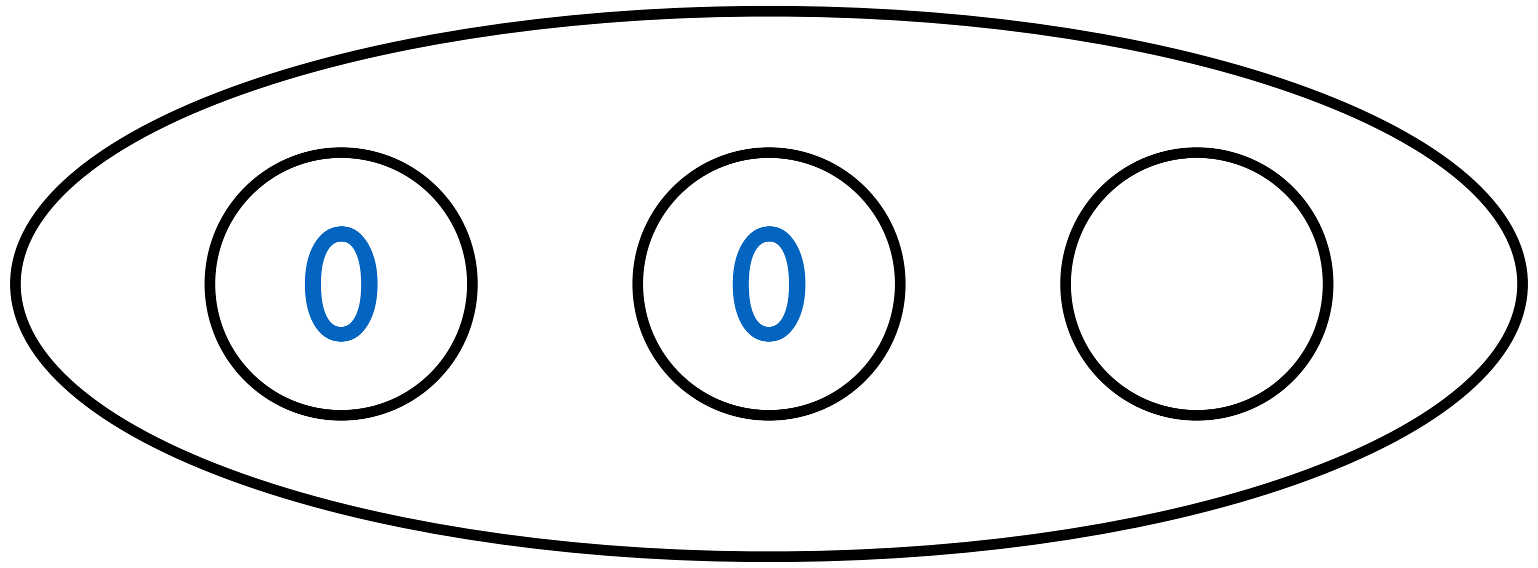

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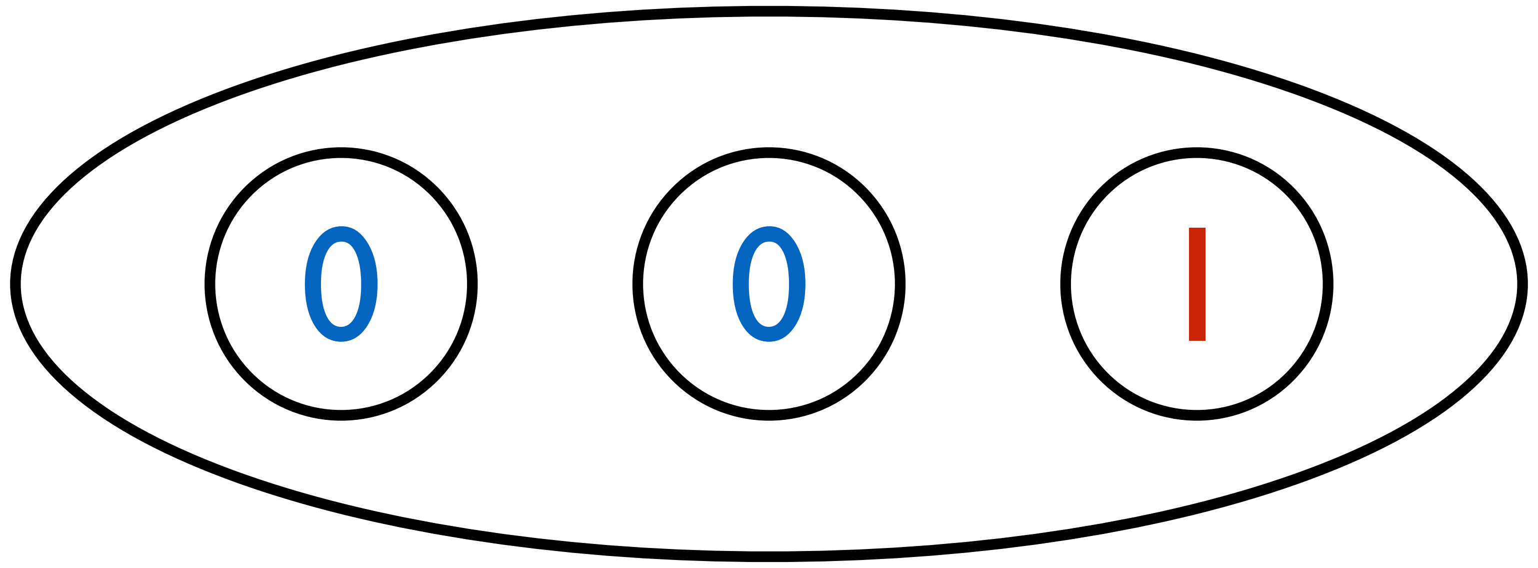

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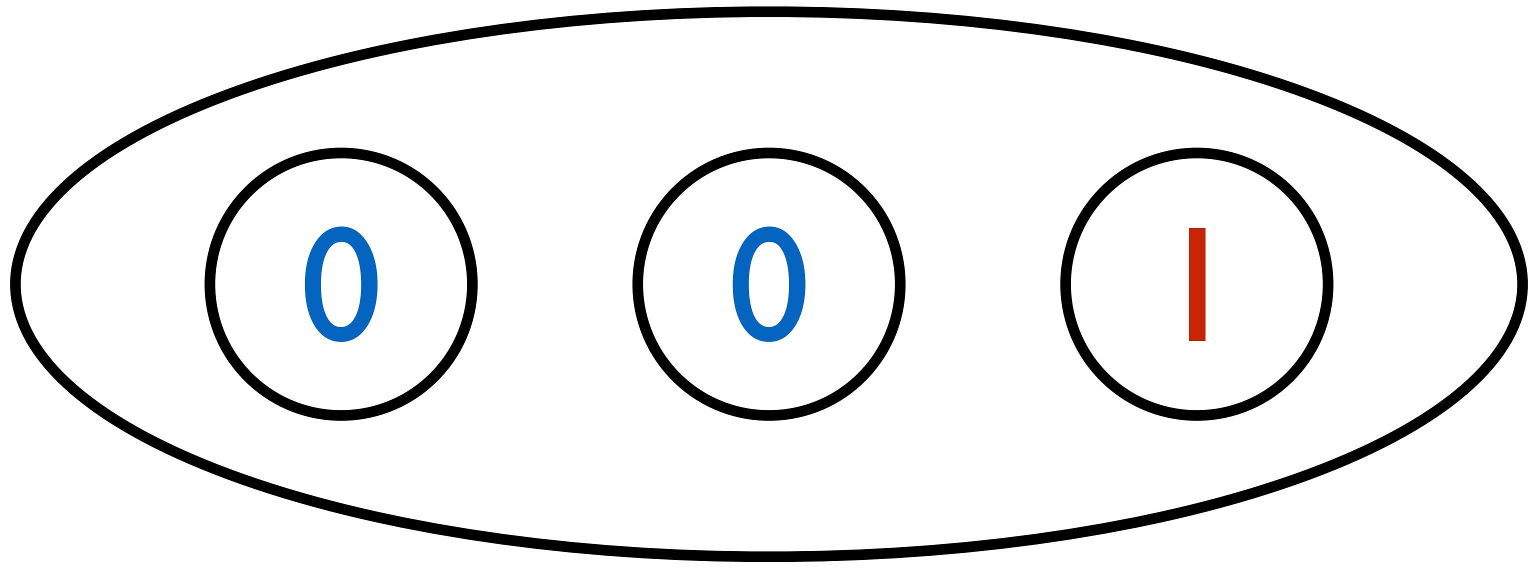
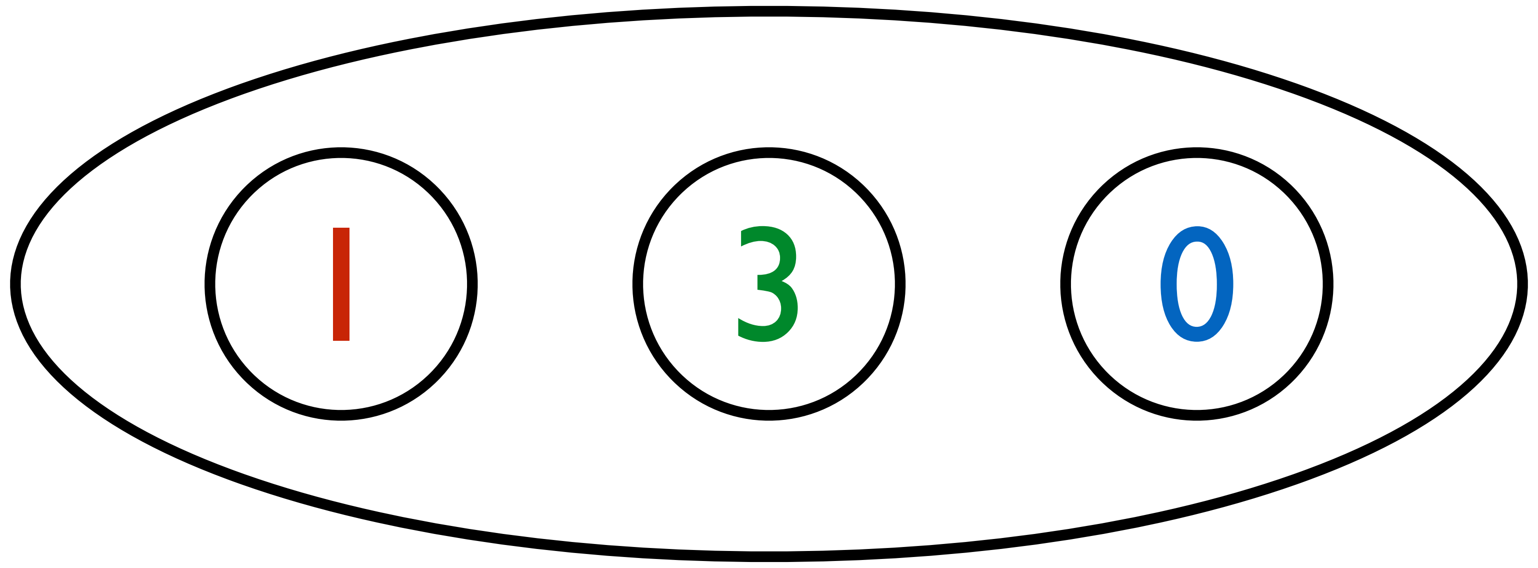


0	0	1
0	1	0
1	0	0

LO Colouring



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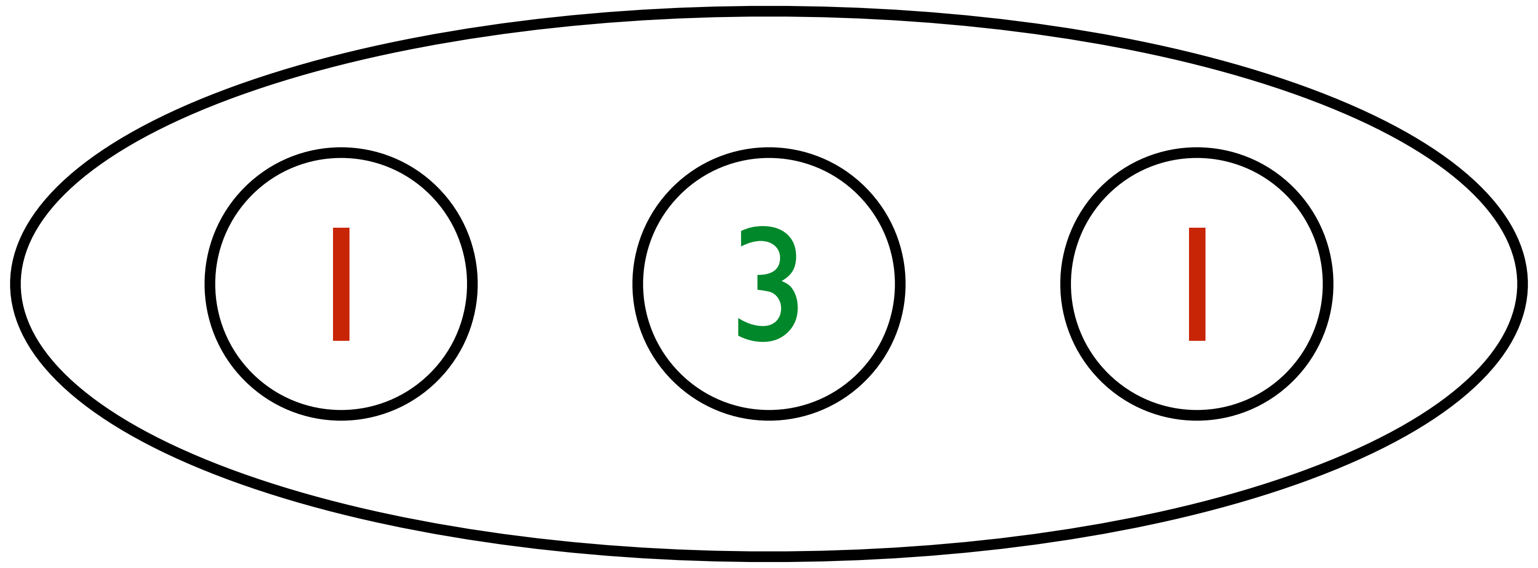
LO Colouring



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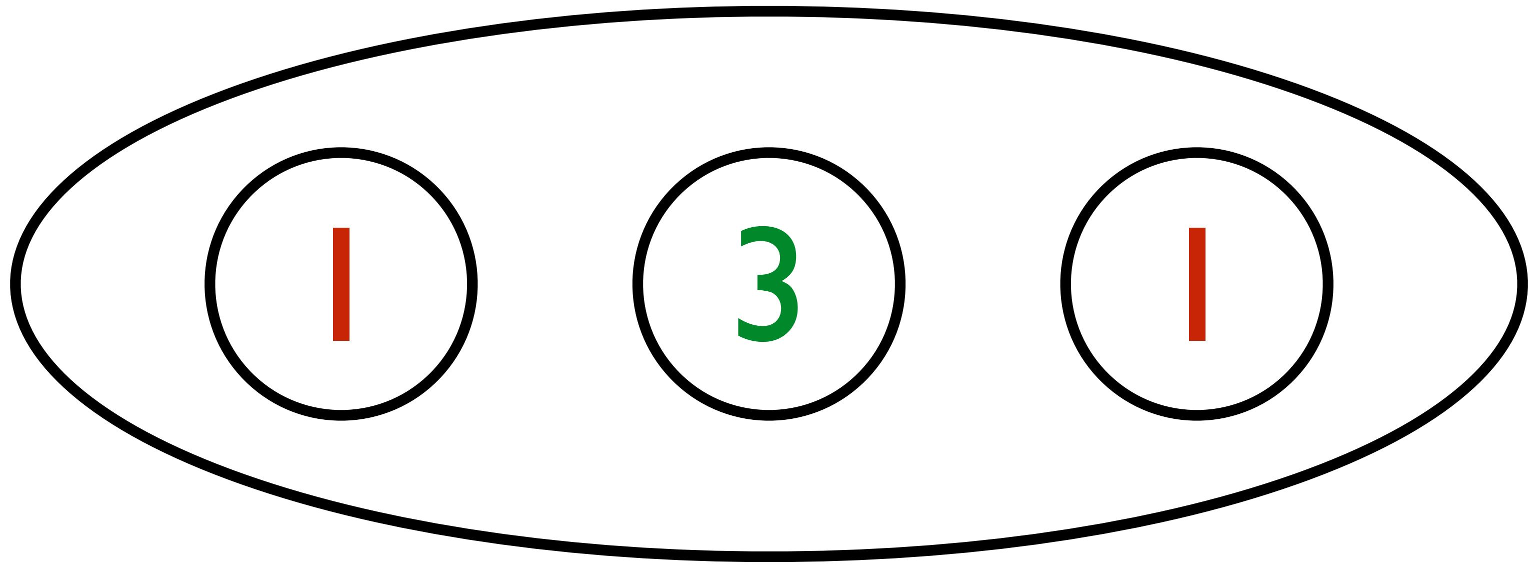

Find a c-LO-colouring of a k-LO-colourable 3-uniform hypergraph.

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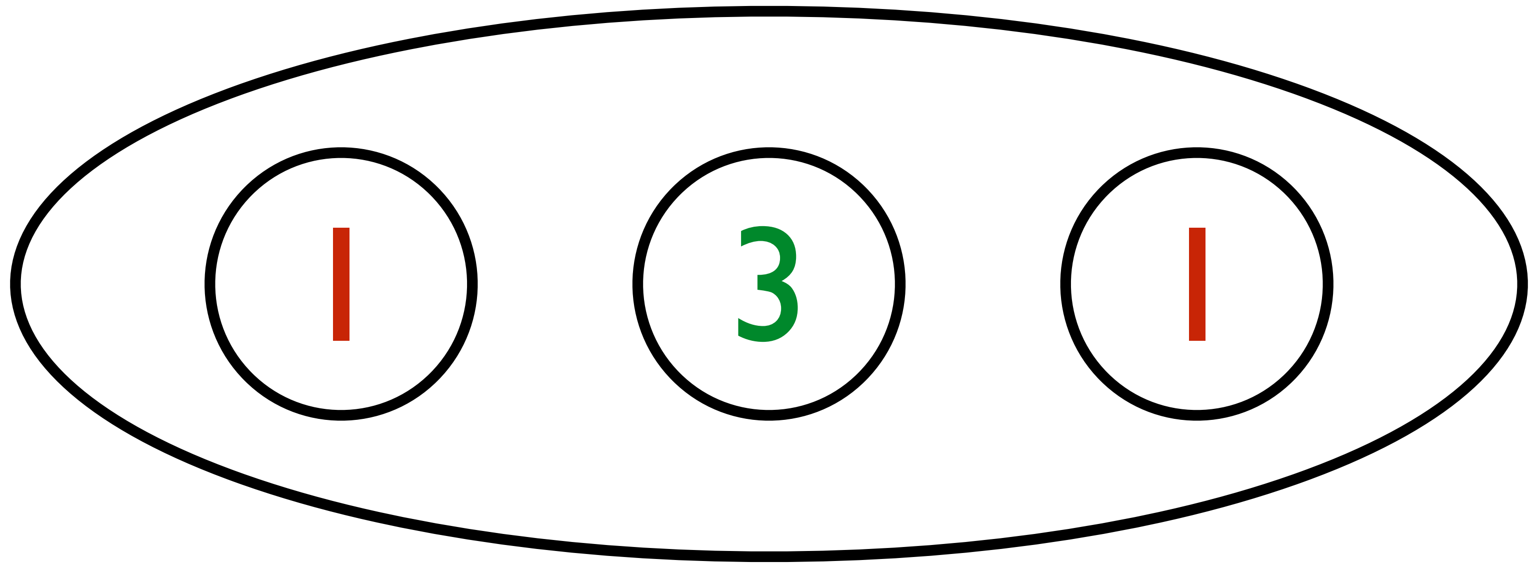

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Conjecture: NP-hard for every constant $c \geq k = 2.$ *[Barto-Battistelli-Berg STACS'21]*

2. CSPs

Constraint Satisfaction Problems

IN: set of variables, set of labels, set of constraints

OUT: assignment that satisfies the given constraints

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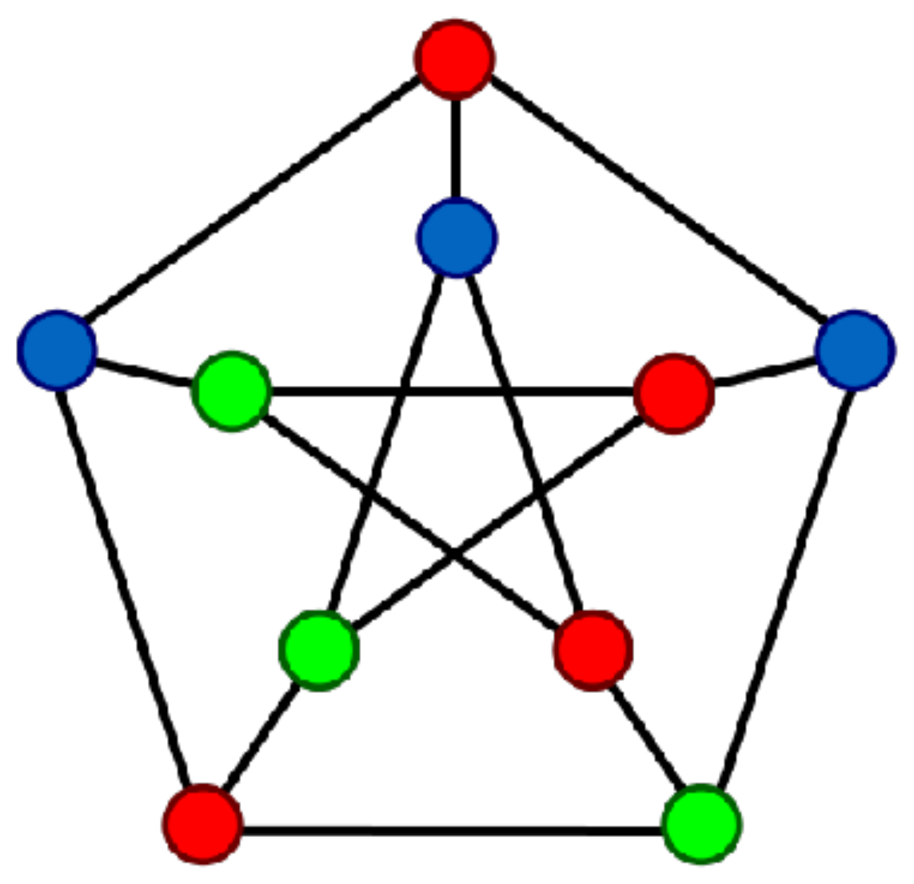
3-Colour



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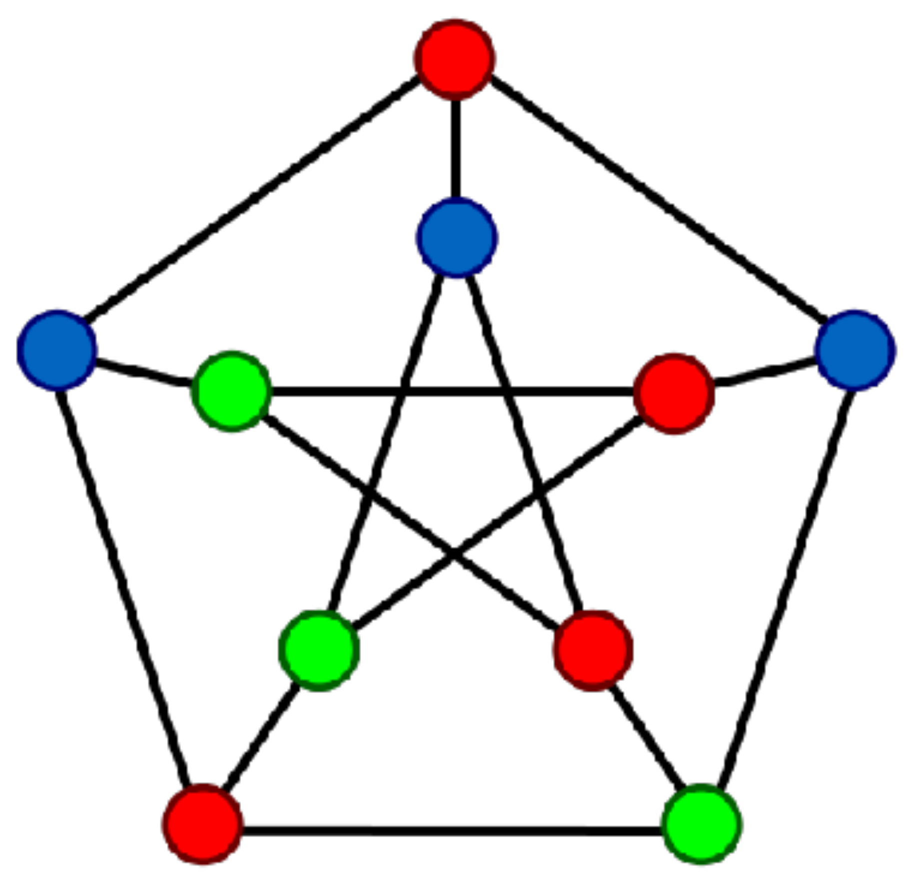

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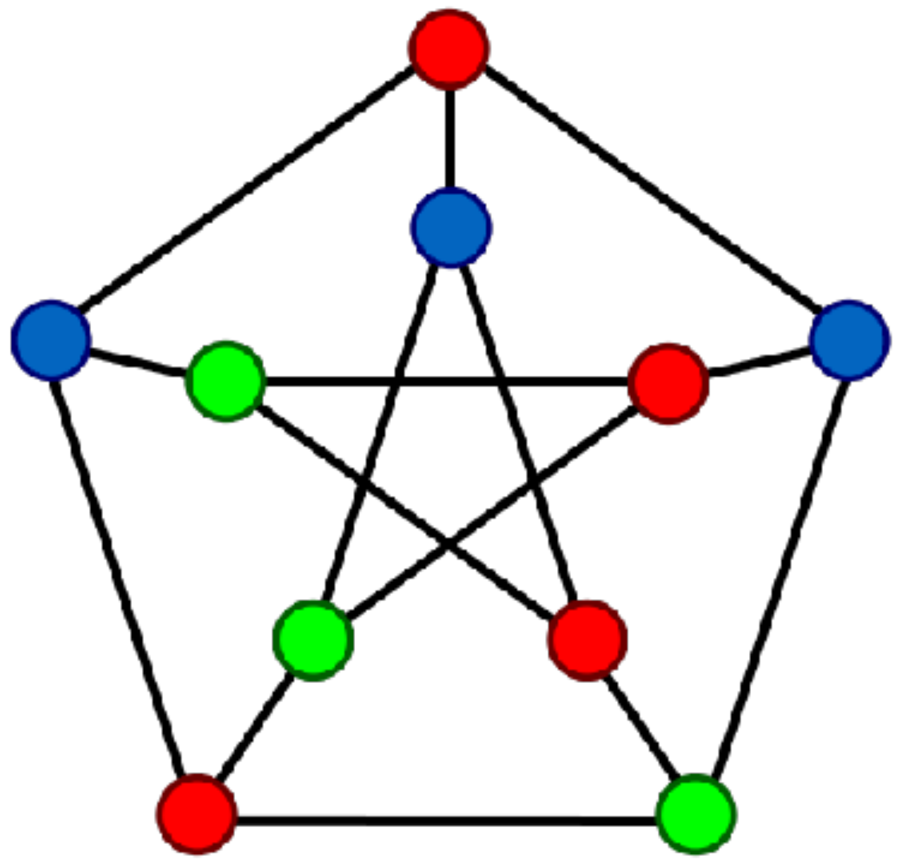

Linear Equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_2 + x_4 + x_5 \qquad &= 0 \\ x_1 + x_3 + x_4 + x_6 &= 2 \end{aligned}$$

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Constraint Satisfaction Problems

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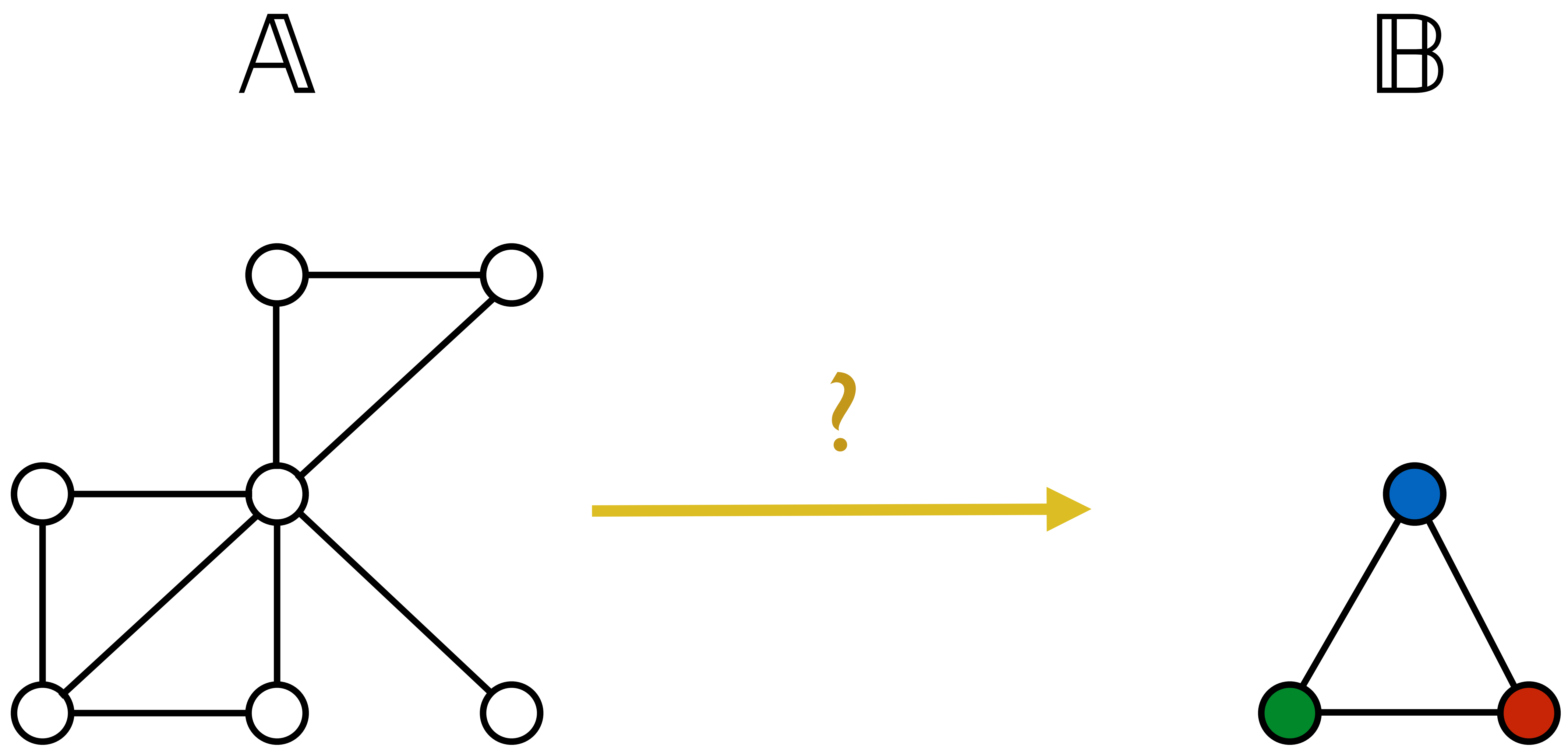
IN: two similar relational structures $\mathbb{A}$ and $\mathbb{B}$

OUT: homomorphism from $\mathbb{A}$ to $\mathbb{B}$

3-Colour



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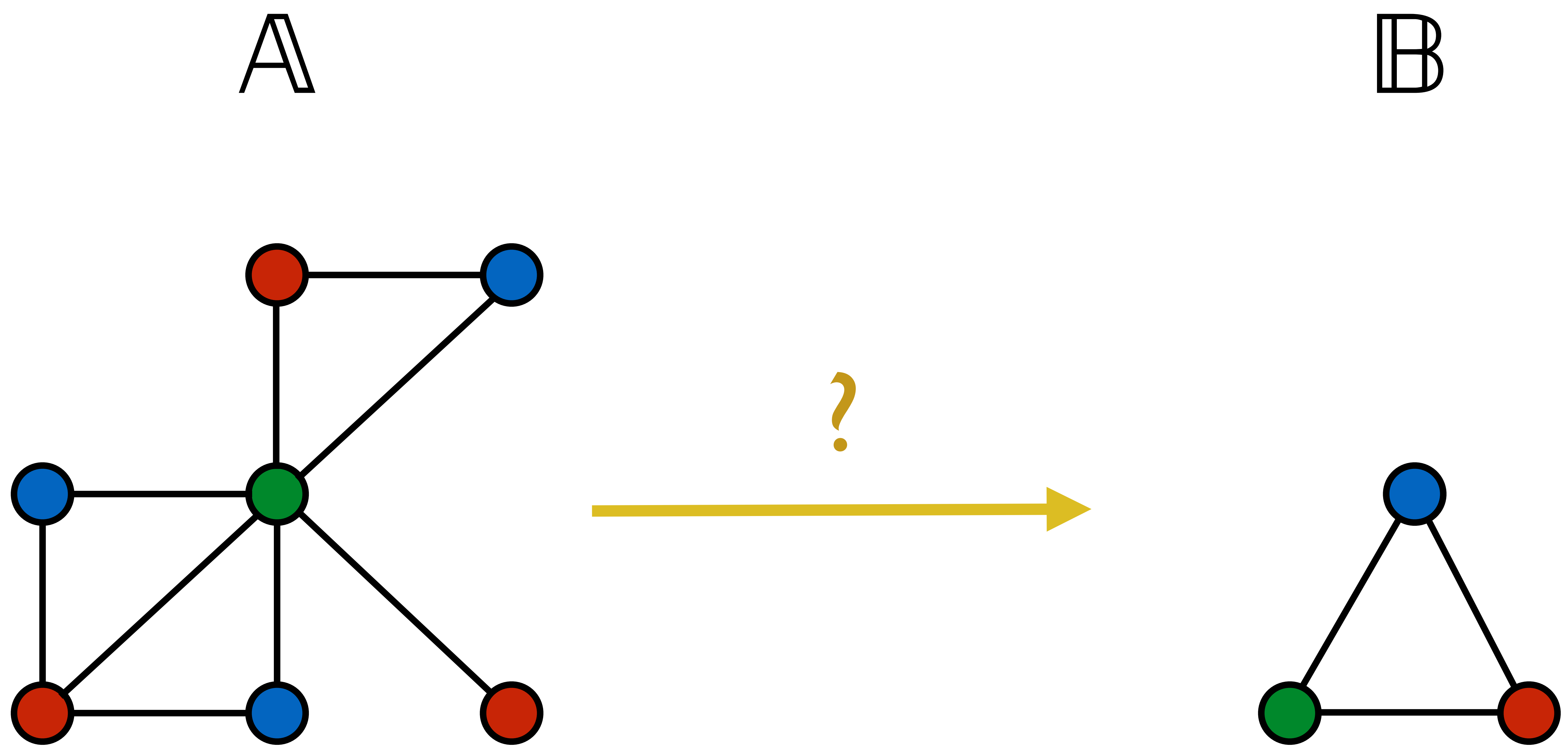

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CSP($\mathbb{B}$)

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CSP($\mathbb{B}$)

Boolean $\mathbb{B}$ *[Schaefer STOC'78]*

graph $\mathbb{B}$ *[Hell-Nešetřil JCTB'90]*

dichotomy conjecture *[Feder-Vardi SICOMP'98]*

⋮

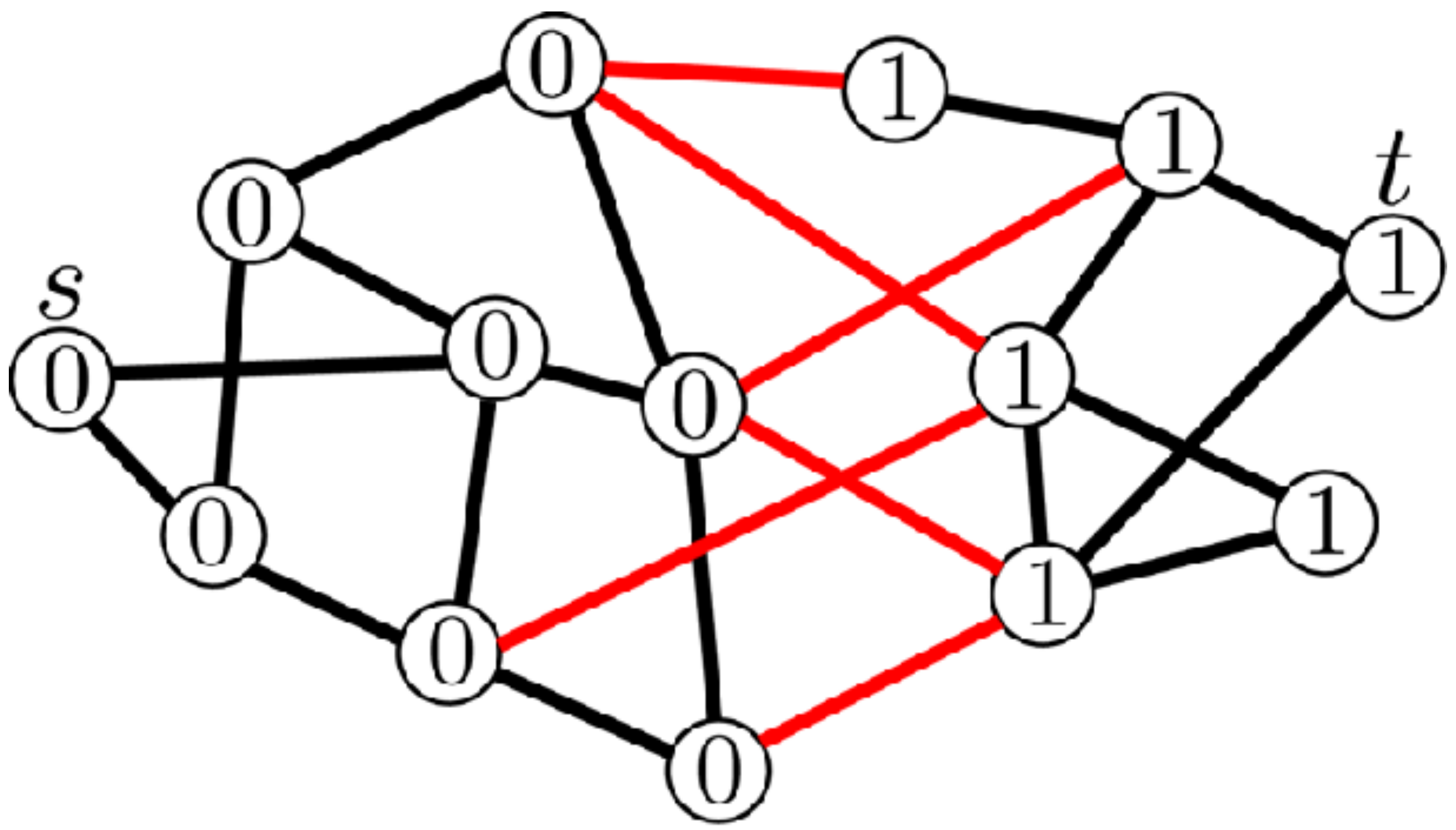
any finite $\mathbb{B}$ *[Bulatov FOCS'17, Zhuk JACM'20]*

CSPs

IN: two similar relational structures $\mathbb{A}$ and $\mathbb{B}$

OUT: homomorphism from $\mathbb{A}$ to $\mathbb{B}$

Max-Cut



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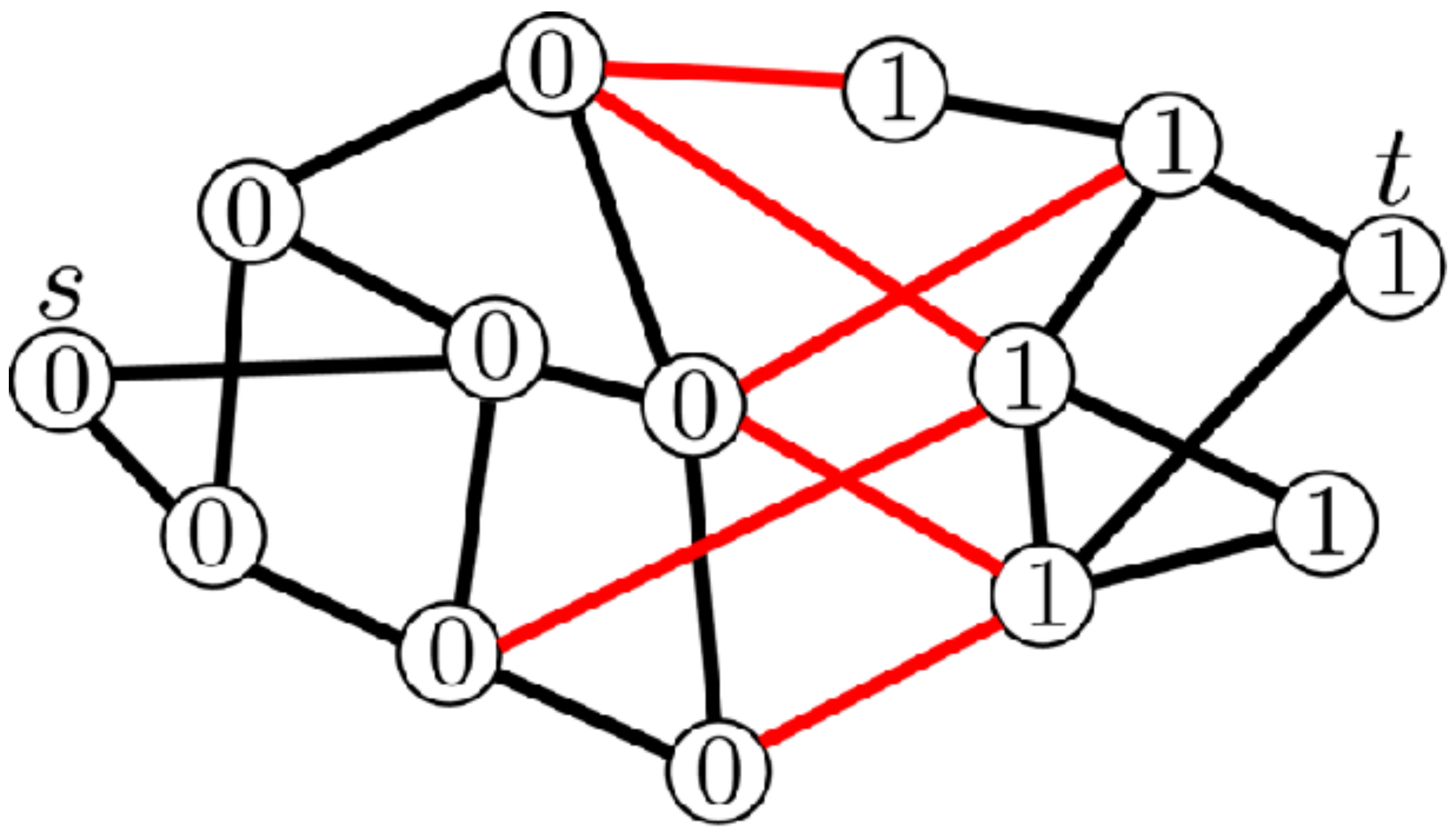
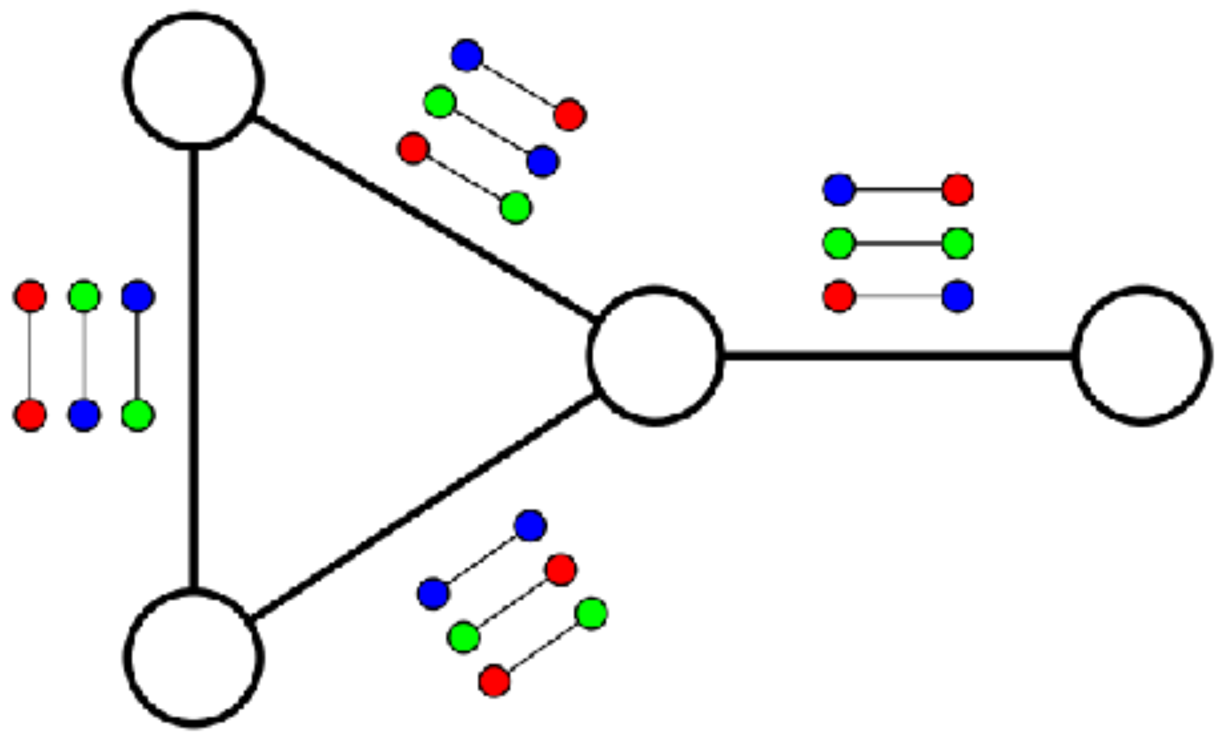


UGC



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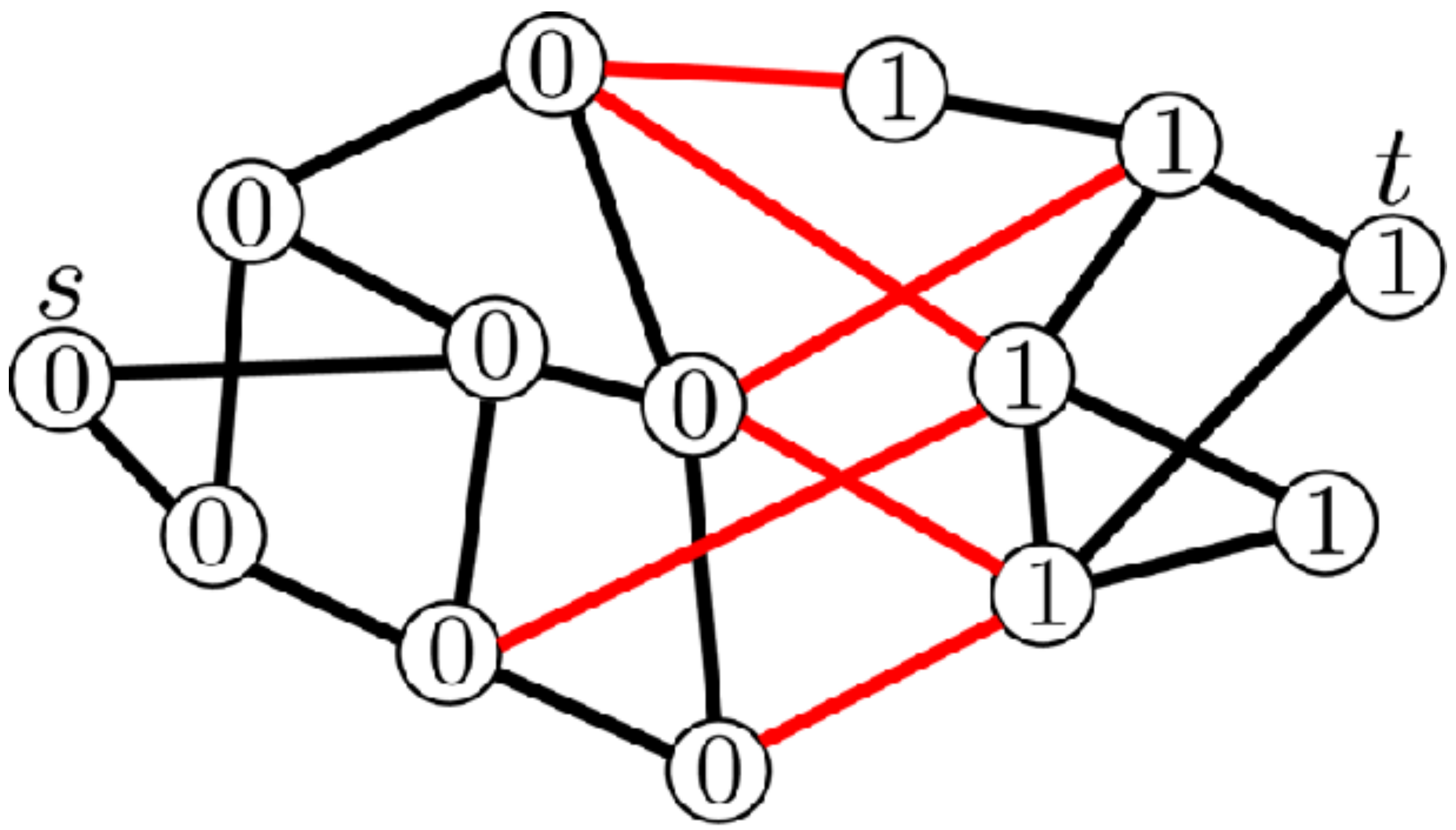
(s,t)-Min-Cut



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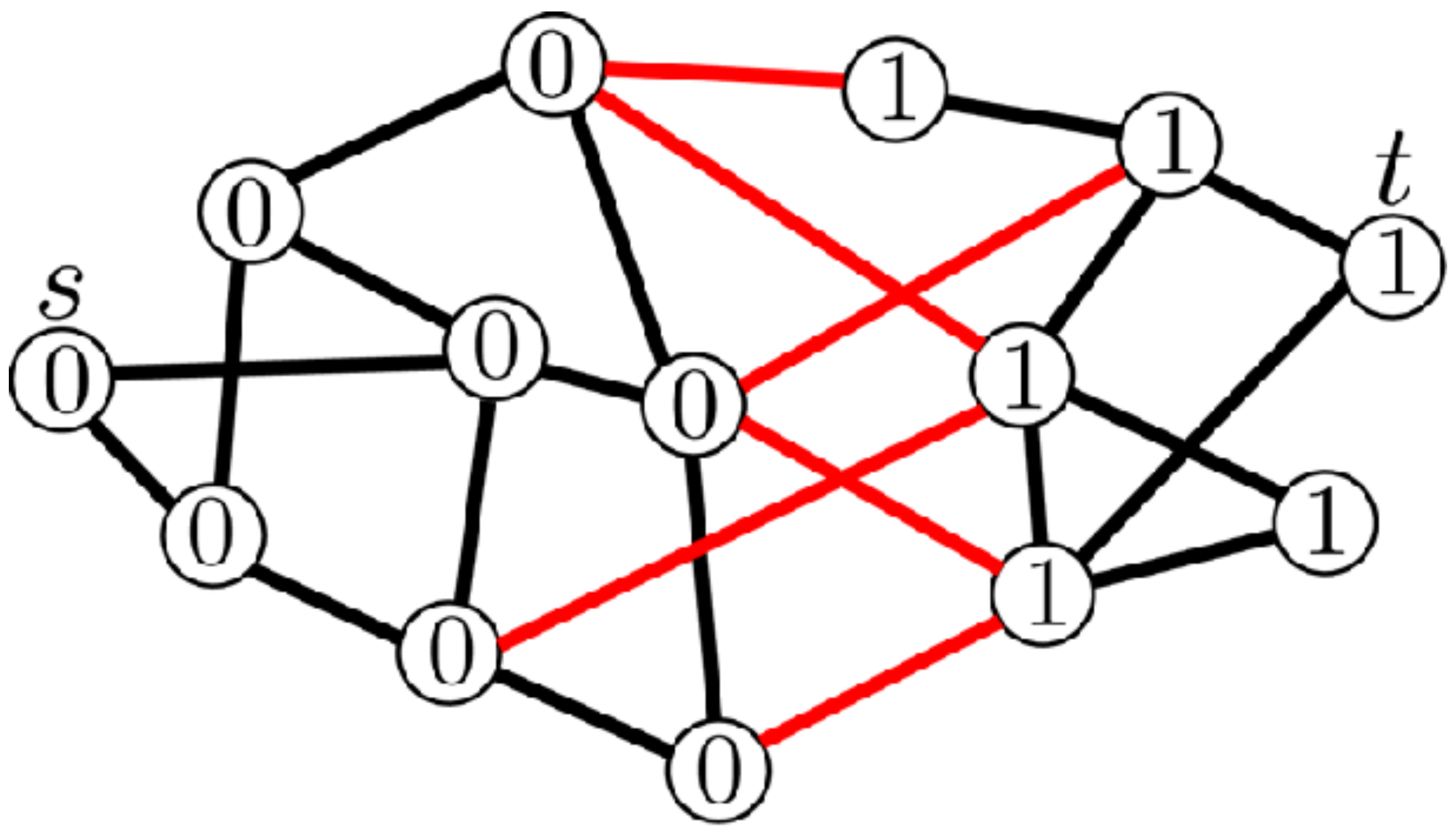
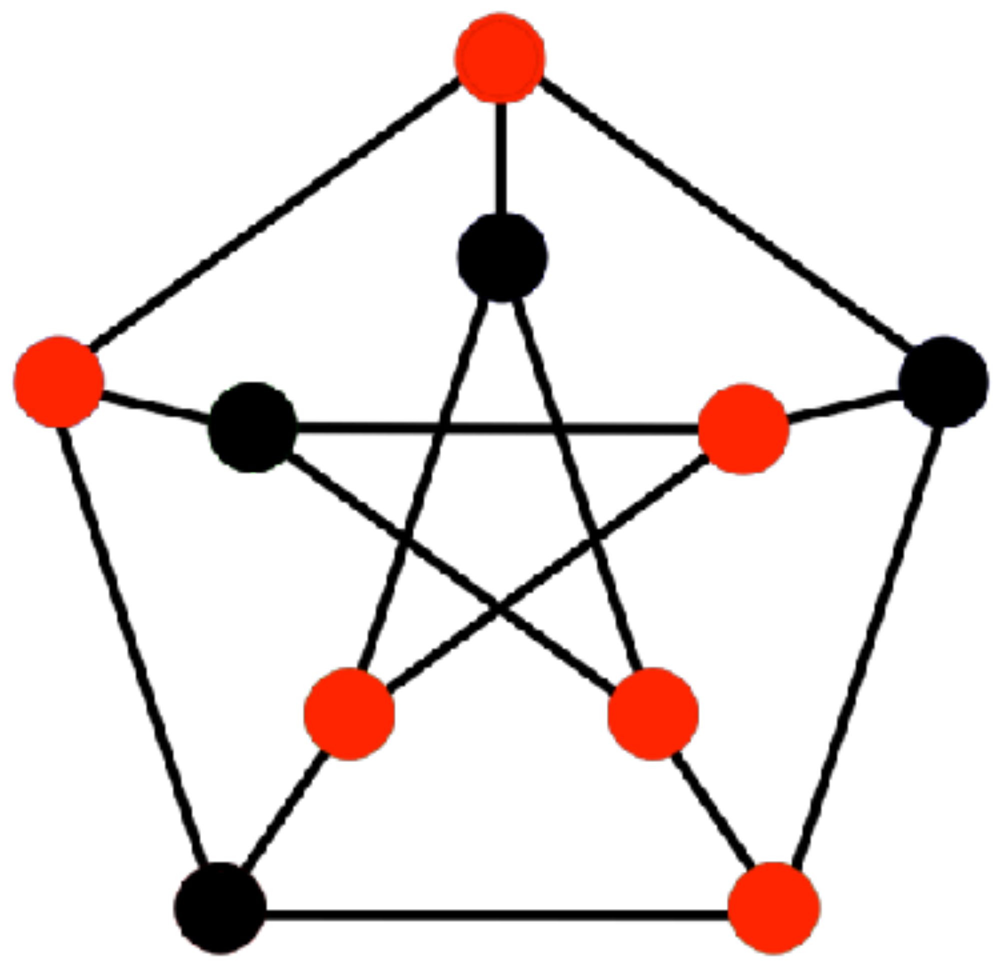
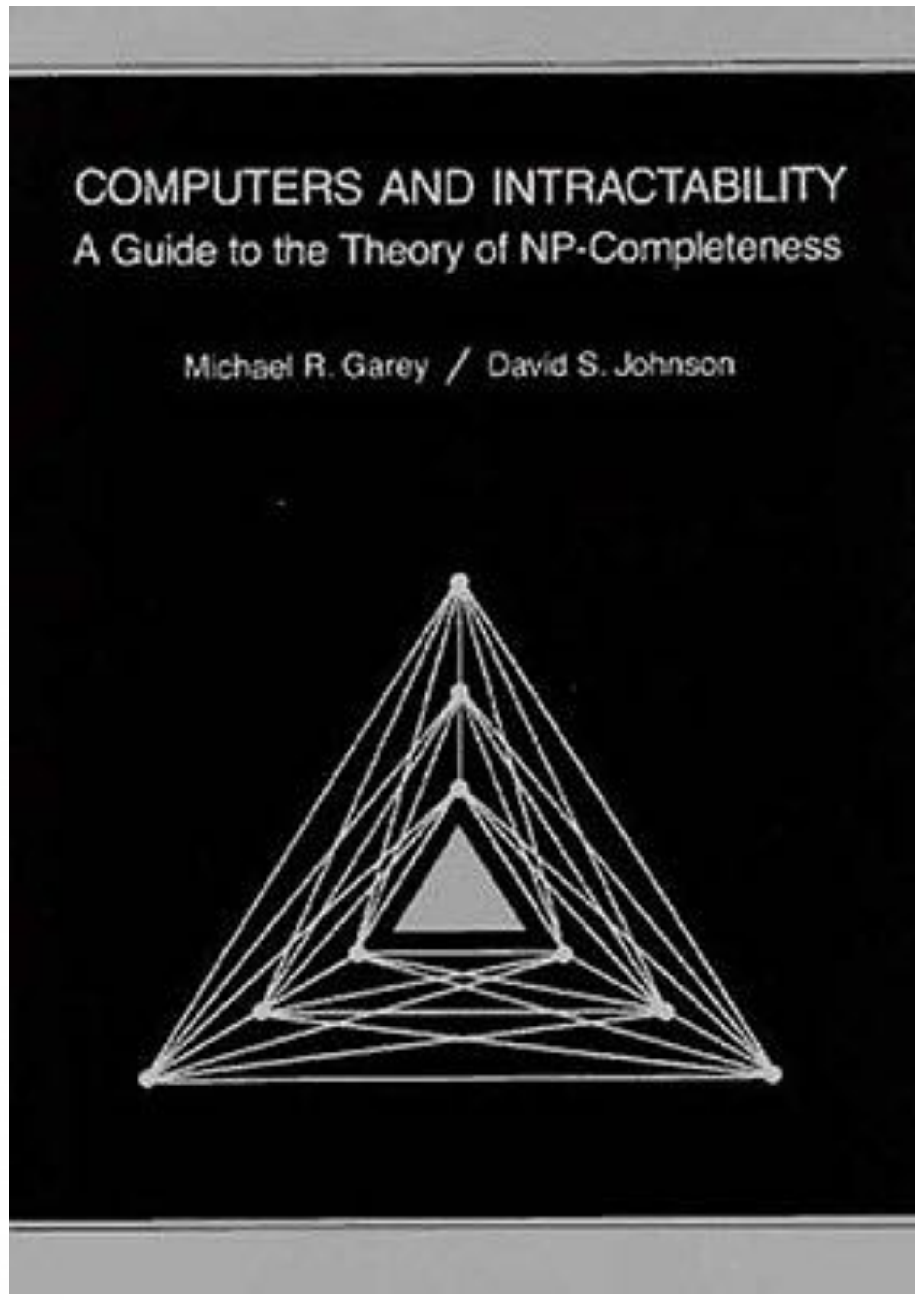


#Covers



How to Relax

“I can’t find an efficient algorithm, but neither can all these famous people.”



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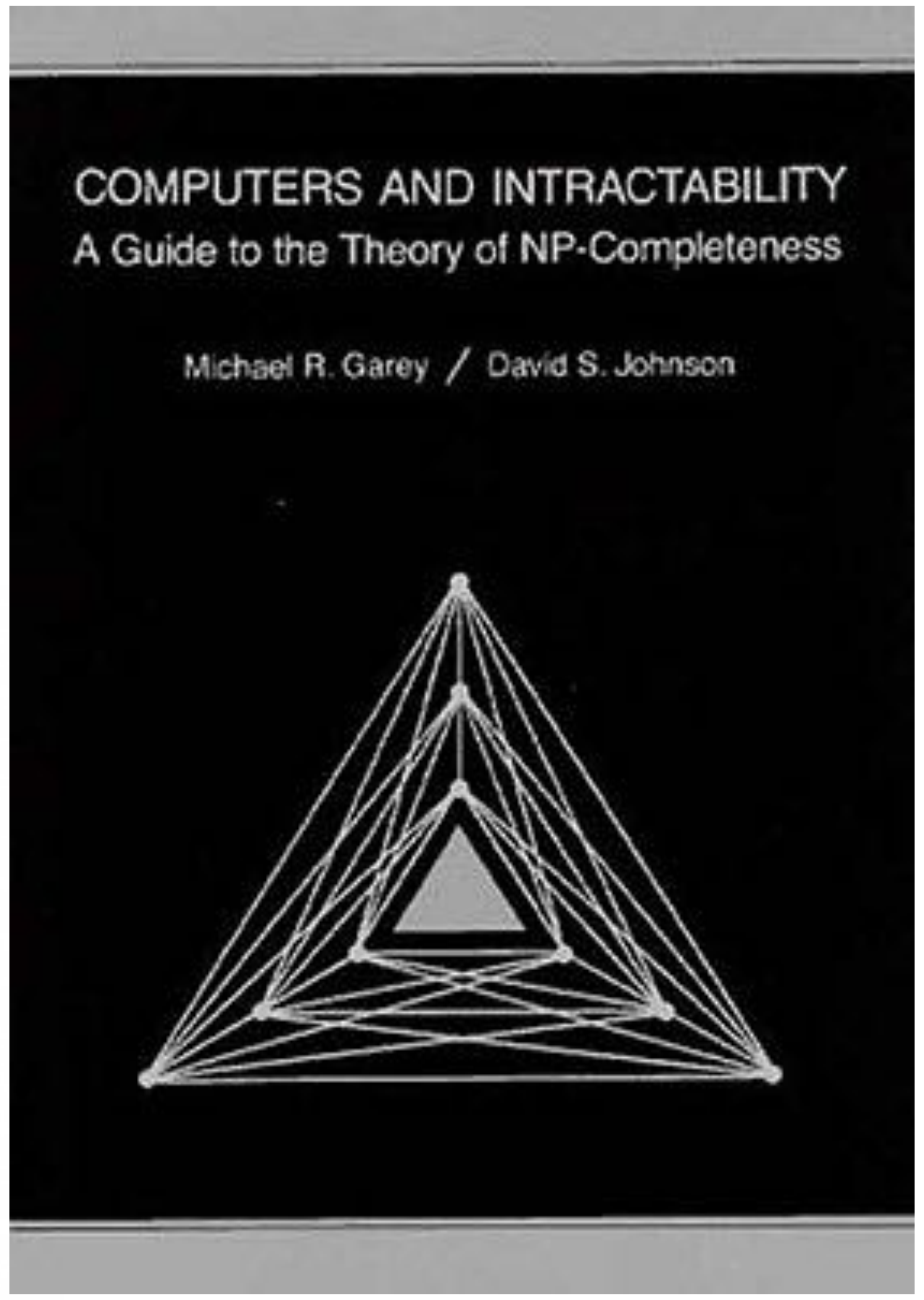
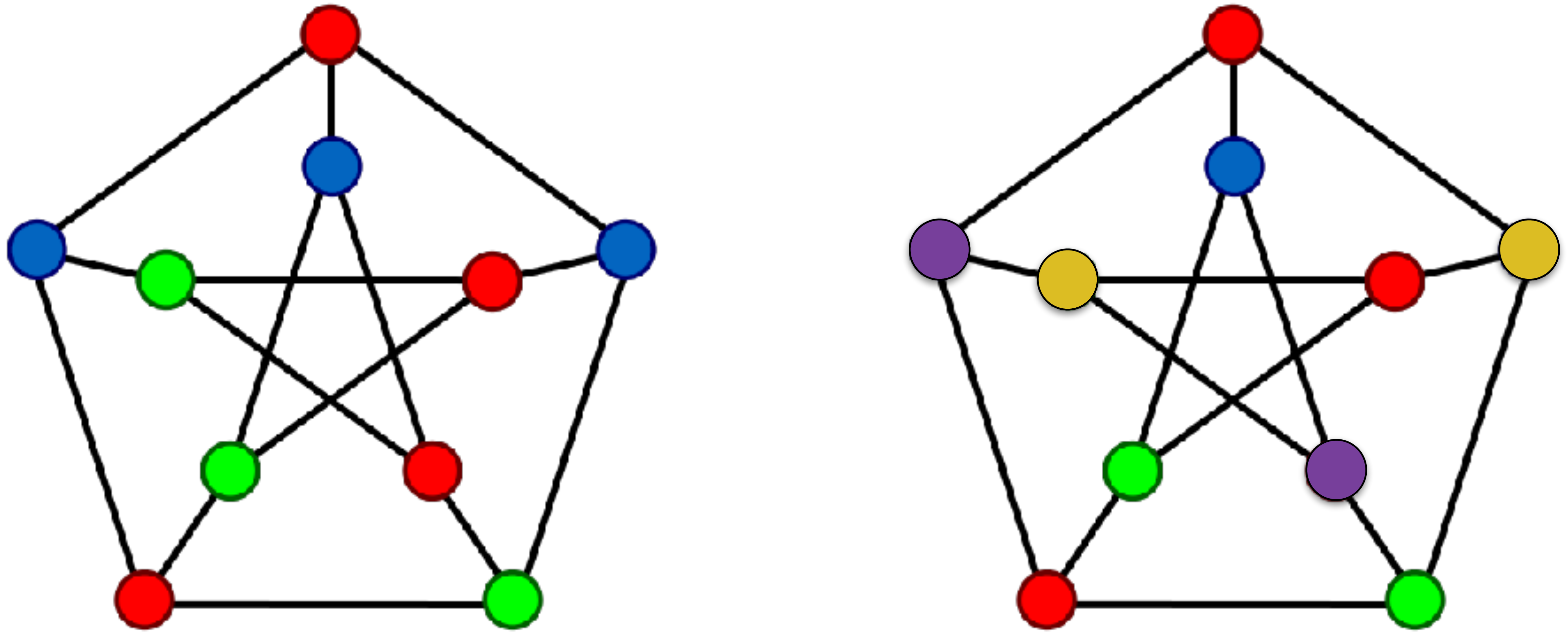


"I can't find an efficient algorithm, but neither can all these famous people."

Satisfy only a fraction of the constraints!

Satisfy a relaxed version of the constraints!

Approximate Graph Colouring



Find a c-colouring of a k-colourable graph ($c \geq k \geq 3$).

Promise CSPs

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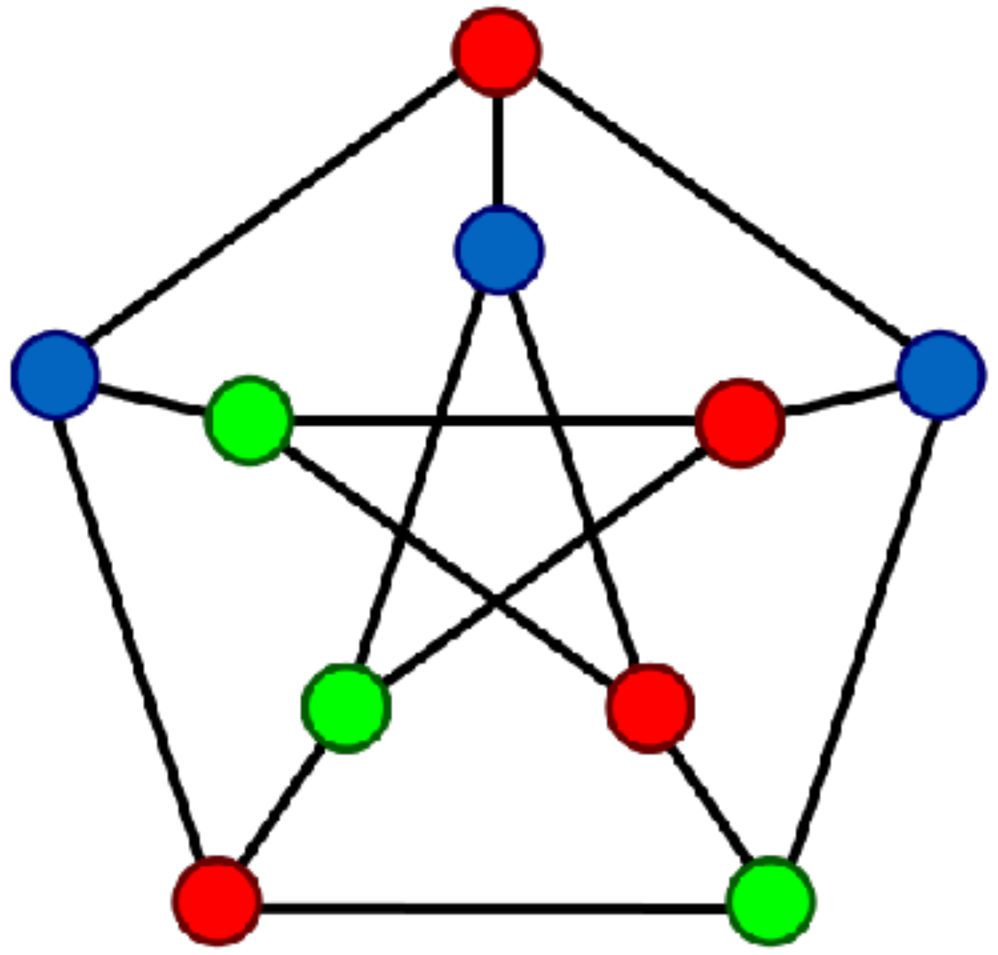
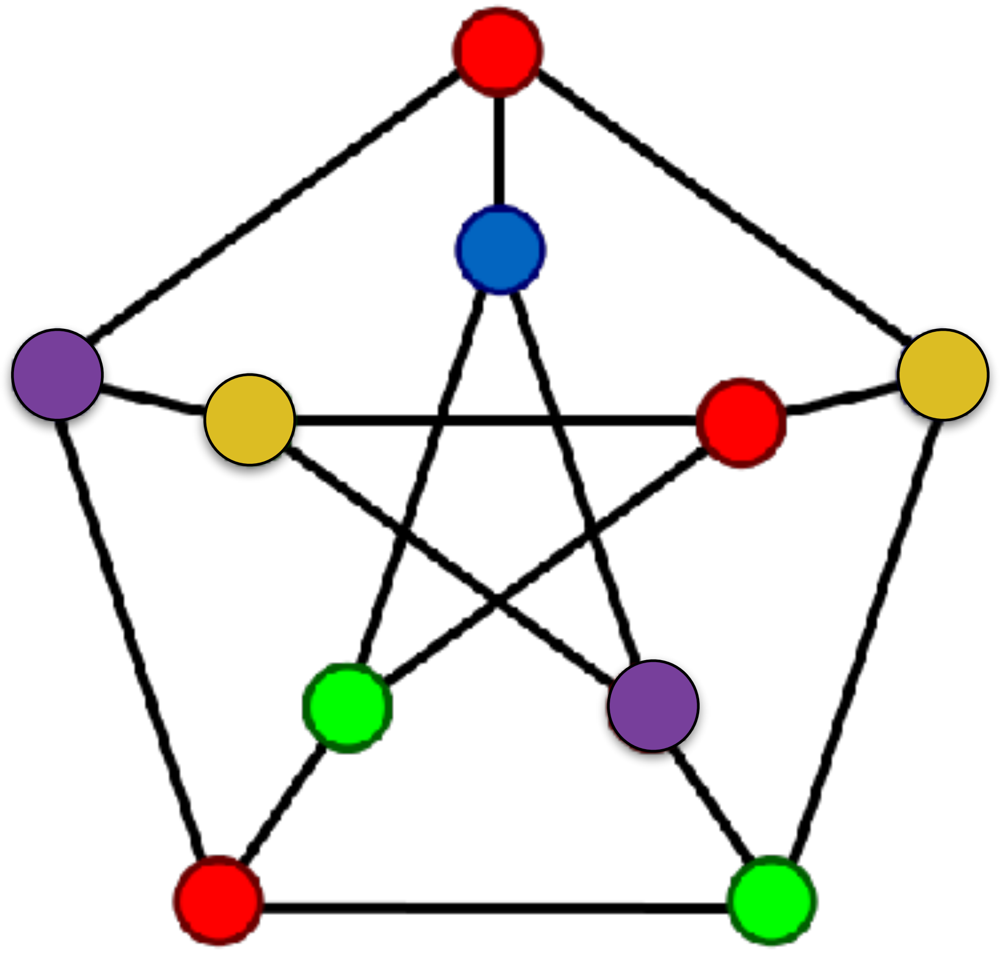


5-Colour



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1-in-3-SAT

(x, y, z)

NAE-3-SAT

1-in-3-SAT	NAE-3-SAT
	$(1,0,0)$
	$(0,1,0)$
$(1,0,0)$	$(0,0,1)$
$(0,1,0)$	$(1,1,0)$
$(0,0,1)$	$(1,0,1)$
	$(0,1,1)$

PCSP($\mathbb{A}$,$\mathbb{B}$)

IN:	set of variables, set of labels, set of strict and weak constraints
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PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \to \mathbb{B}$

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PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \to \mathbb{B}$

IN: set of variables, set of labels, set of strict and weak constraints

PROMISE: exists assignment using strict constraints

OUT: assignment using weak constraints

IN: ▯

PROMISE: exists homomorphism from ▯ to $\mathbb{A}$

OUT: homomorphism from ▯ to $\mathbb{B}$

PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \to \mathbb{B}$

IN: set of variables, set of labels,
set of strict and weak constraints

PROMISE: exists assignment using strict constraints

OUT: assignment using weak constraints

search

IN: ▯

PROMISE: exists homomorphism from ▯ to $\mathbb{A}$

OUT: homomorphism from ▯ to $\mathbb{B}$

PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \rightarrow \mathbb{B}$

search

IN: ▯

PROMISE: exists homomorphism from ▯ to $\mathbb{A}$

OUT: homomorphism from ▯ to $\mathbb{B}$

PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \to \mathbb{B}$

decision

IN: ▯

OUT YES: if there is homomorphism from ▯ to $\mathbb{A}$

OUT NO: if there is no homomorphism from ▯ to $\mathbb{B}$

search

IN: ▯

PROMISE: exists homomorphism from ▯ to $\mathbb{A}$

OUT: homomorphism from ▯ to $\mathbb{B}$

PCSP($\mathbb{A}$,$\mathbb{B}$)

$\mathbb{A} \to \mathbb{B}$

decision

IN: ▯

OUT YES: if there is homomorphism from ▯ to $\mathbb{A}$

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$\leq_p$

search

IN: ▯

PROMISE: exists homomorphism from ▯ to $\mathbb{A}$

OUT: homomorphism from ▯ to $\mathbb{B}$

PCSP($\mathbb{A}$,$\mathbb{B}$)

- PCSP($\mathbb{A}$,$\mathbb{A}$) = CSP($\mathbb{A}$)
- PCSP($\mathbb{A}$,$\mathbb{B}$) in P if CSP($\mathbb{A}$) in P or CSP($\mathbb{B}$) in P
- PCSP($\mathbb{A}$,$\mathbb{B}$):
  - aproximability of CSP($\mathbb{A}$) on satisfiable instances
  - CSP($\mathbb{B}$) with restricted instances

3. Results

SetSAT

Thm: (1,g,k)-SAT in P if $\frac{g}{k} \geq \frac{1}{2}$ and NP-hard otherwise. *[Austrin-Guruswami-Håstad SICOMP'17]*

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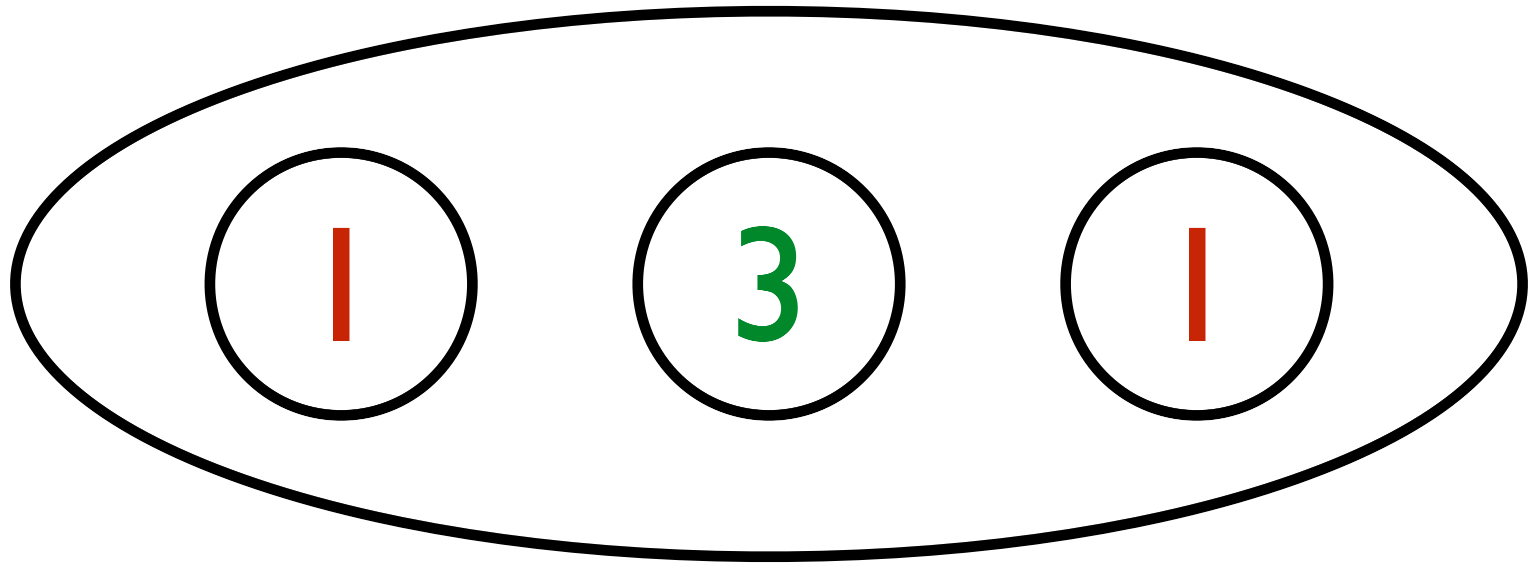
Thm: (1,g,k)-SAT in P if $\frac{g}{k} \geq \frac{1}{2}$ and NP-hard otherwise. *[Austrin-Guruswami-Håstad SICOMP'17]*

Thm: (1,g,k)-SetSAT in P if $\frac{g}{k} \geq \frac{s}{s+1}$ and NP-hard otherwise. *[Brandts-Wrochna-Živný ACM ToCT'21]*

LO Colouring



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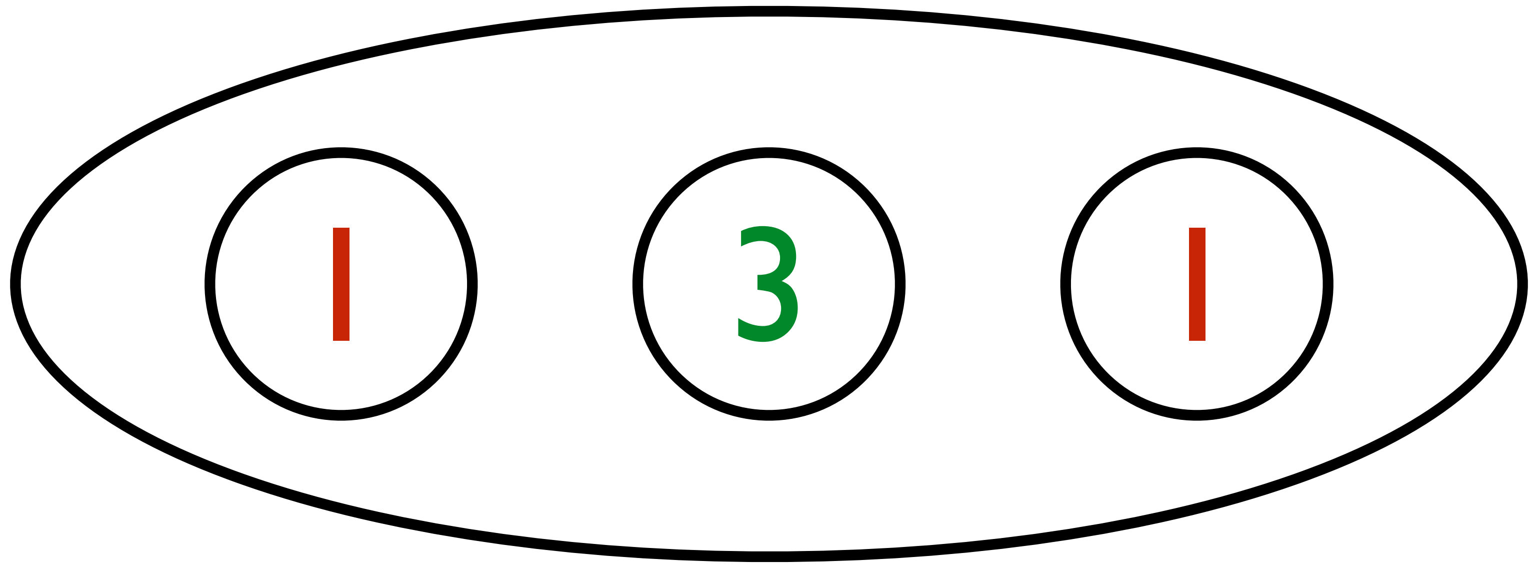
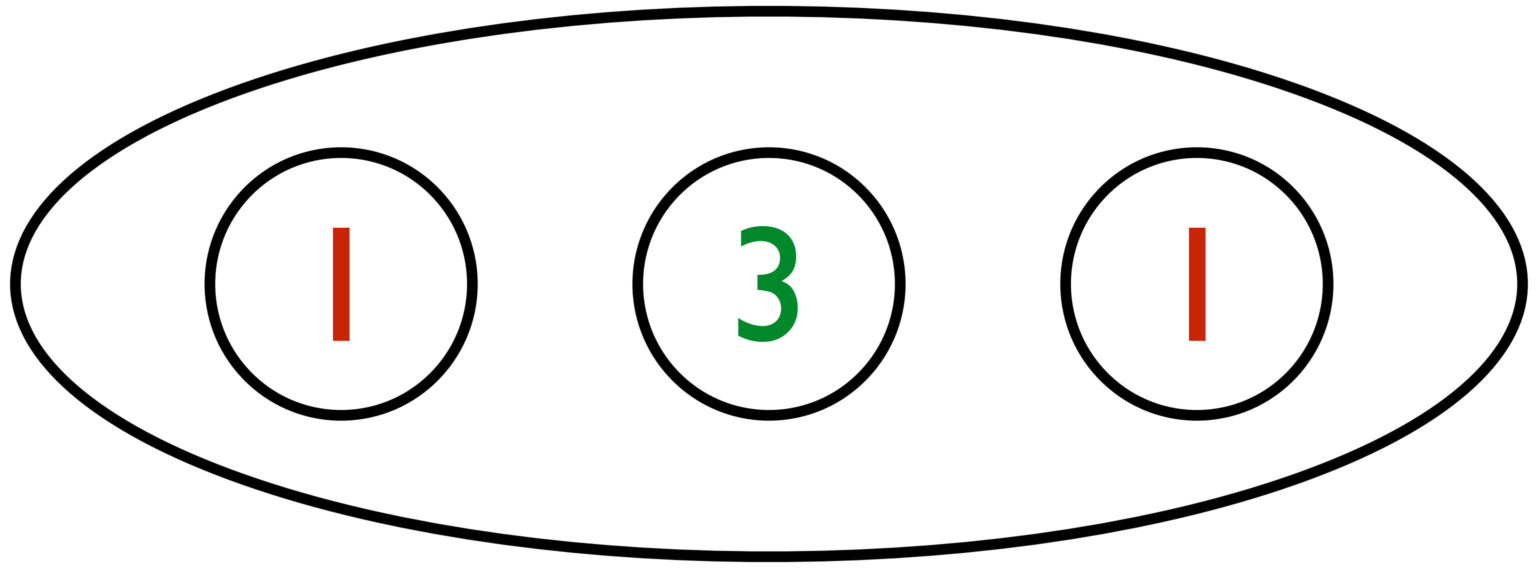


unique maximum

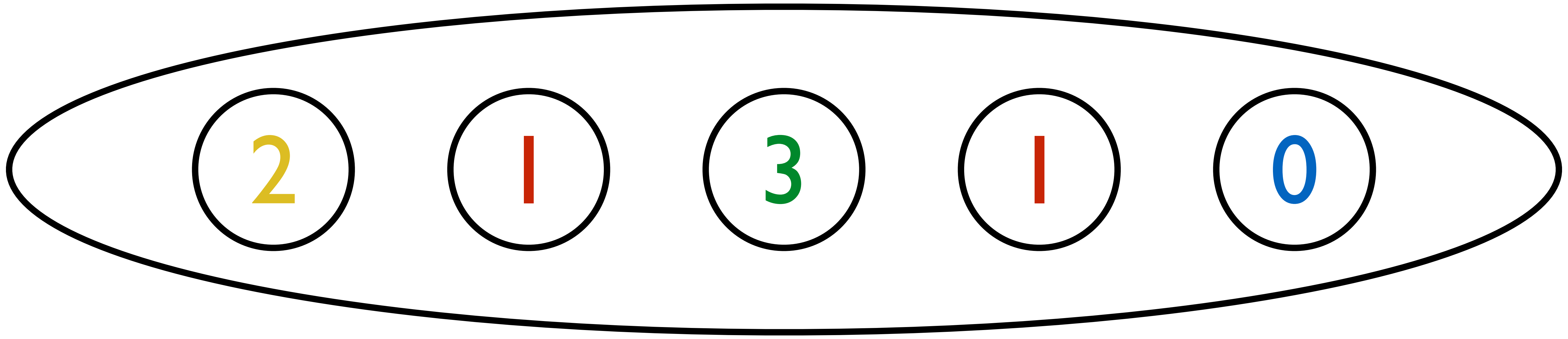
LO Colouring



unique maximum

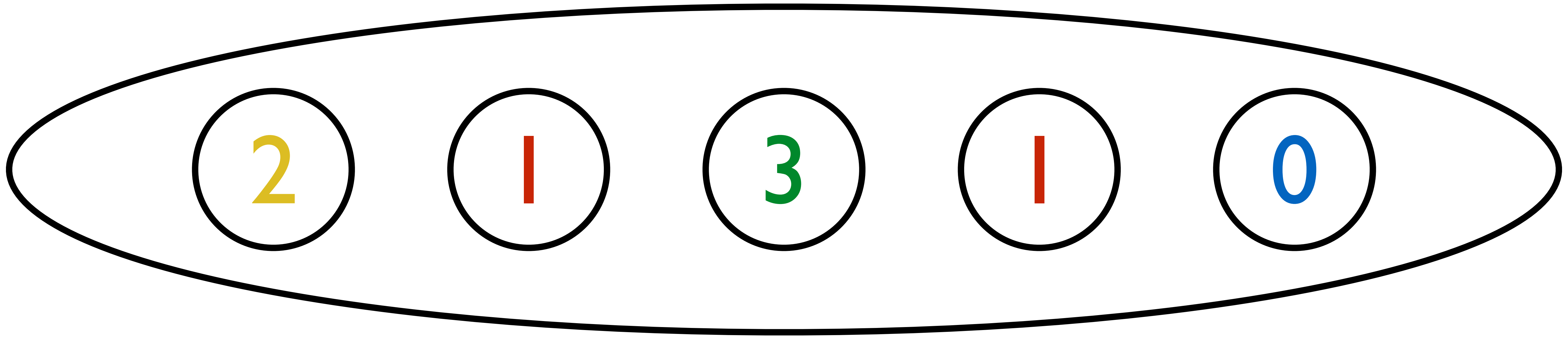
Problem: Complexity of PCSP(LO_2, LO_c)? *[Barto-Battistelli-Berg STACS'21]*

LO Colouring



unique maximum

LO Colouring



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Thm: *[Nakajima-Živný ACM ToCT'22]*

- PCSP(LO_k^r, LO_c^r) is NP-hard for all $2 \leq k \leq c$ and $r \geq c - k + 4$.

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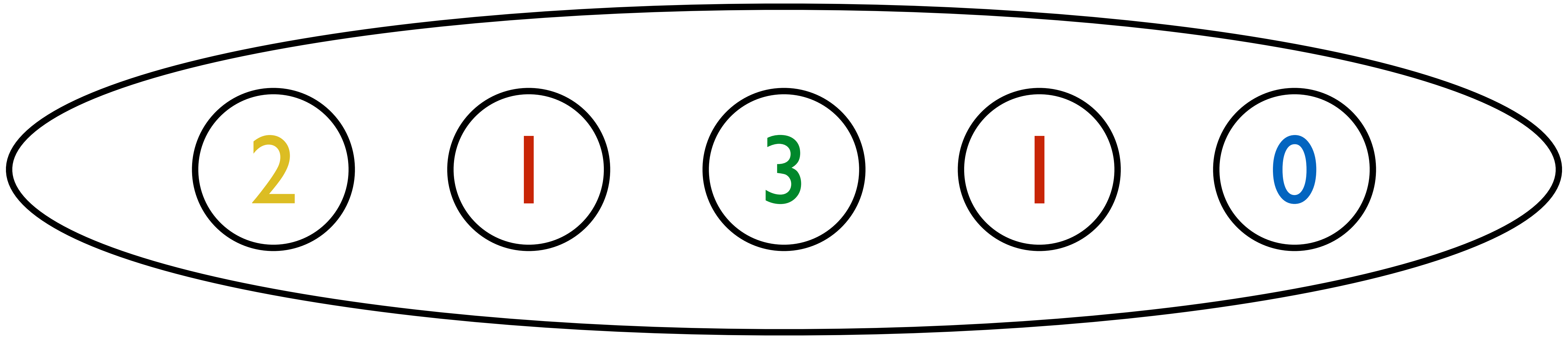

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- PCSP(LO_2^3, LO_ℓ^3) in P for $\ell = O(\sqrt[3]{n \log \log n / \log n})$.

PCSP(K_3,K_5)

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Thm: PCSP(K_3,K_5) is NP-hard. *[Barto-Bulín-Krokhin-Opršal JACM'21]*

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PCSP(K_k,K_c) with $c = k$. *[Karp CCC'72]*

PCSP(K_k,K_c) with $c = k + 2\lfloor k/3 \rfloor - 1$. *[Khanna-Linial-Safra Comb.'00]*

PCSP(K_k,K_c) with $c = 2k - 5$ and $k \geq 6$. *[Garey-JohsonJ JACM'76]*

PCSP(K_k,K_c) with $c = 2k - 2$. *[Brakensiek-Guruswami CCC'16]*

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K3,K4

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Thm: PCSP(K_k,K_c) is NP-hard with c $\approx 2^k$ and $k \geq 4$. *[Krokhin-Opršal-Wrochna-Živný SICOMP'23]*

IP

IP

PCSP($\mathbb{A}$,$\mathbb{B}$) instance $\mathbb{I}$

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PCSP(𝔸,𝔹) instance $\mathbb{I}$

$$\lambda_{\mathbf{x},R}(\mathbf{a}) \in \{0,1\} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbb{I}}$$

IP

PCSP($\mathbb{A}$,$\mathbb{B}$) instance $\mathbb{I}$

$$\sum_{\mathbf{a} \in R^{\mathbb{A}}} \lambda_{\mathbf{x},R}(\mathbf{a}) = 1 \qquad\qquad R \in \sigma, \mathbf{x} \in R^{\mathbb{I}}$$

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BLP

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$$\sum_{\mathbf{a} \in R^{\mathsf{A}}, a_i = a} \lambda_{\mathbf{x},R}(\mathbf{a}) = \lambda_{x_i,R_u}(a) \qquad a \in A, i \in [\mathrm{ar}(R)]$$

$$\lambda_{\mathbf{x},R}(\mathbf{a}) \in [0,1] \qquad R \in \sigma, \mathbf{x} \in R^{\mathbb{I}}$$

AIP

PCSP(𝔸,𝔹) instance 𝕂

$$\sum_{\mathbf{a}\in R^{\mathbb{A}}} \lambda_{\mathbf{x},R}(\mathbf{a}) = 1 \qquad R \in \sigma, \mathbf{x} \in R^{\mathbb{K}}$$

$$\sum_{\mathbf{a}\in R^{\mathsf{A}}, a_i = a} \lambda_{\mathbf{x},R}(\mathbf{a}) = \lambda_{x_i,R_u}(a) \qquad a \in A, i \in [\mathrm{ar}(R)]$$

$$\lambda_{\mathbf{x},R}(\mathbf{a}) \in \mathbb{Z} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbb{K}}$$

BA

Run BLP, ignore assignments getting 0, run AIP.

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Problem: Power of BA^k?

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Problem: Power of BA^k?

Thm: AGC not solved by BA^k. *[Ciardo-Živný STOC'23]*