Generating maximal feasible solutions for a chance-constrained knapsack inequality

Khaled Elbassioni

Khalifa University of Science and Technology, Abu Dhabi, UAE

khaled.elbassioni@ku.ac.ae

September 25, 2023

Chance-constrained knapsack inequality

- Consider the knapsack problem:

$$\sum_j a_j x_j \leq \; c$$
$$\mathbf{x} \in \{0,1\}^n$$

where $a_j \geq 0$

- The elements of $[n] := \{1, \ldots, n\}$ can be interpreted as items to be packed into a knapsack of capacity c, where a_j represents the size requirement of item j

Chance-constrained knapsack inequality

- Consider the knapsack problem:

$$\sum_j a_j x_j \leq c$$
$$\mathbf{x} \in \{0,1\}^n$$

where $a_j \geq 0$

- The elements of $[n] := \{1, \ldots, n\}$ can be interpreted as items to be packed into a knapsack of capacity c, where a_j represents the size requirement of item j
- In the *stochastic* version, the vector $\mathbf{a} = (a_j \mid j \in [n])$ is drawn from a *multivariate normal distribution* with mean $\bar{\mathbf{a}} \in \mathbb{R}^n_+$ and covariance matrix $\Sigma \succeq 0$, i.e., $\mathbf{a} \sim N(\bar{\mathbf{a}}, \Sigma)$

Chance-constrained knapsack inequality

- Given $\mathbf{a} \sim N(\bar{\mathbf{a}}, \Sigma)$ and $\alpha \in [0,1]$, a chance-constrained knapsack inequality can be written as

$$\Pr[\mathbf{a}^\top \mathbf{x} \leq \; c] \geq \alpha$$
$$\mathbf{x} \in \{0,1\}^n$$

Chance-constrained knapsack inequality

- Given $\mathbf{a} \sim N(\bar{\mathbf{a}}, \Sigma)$ and $\alpha \in [0,1]$, a chance-constrained knapsack inequality can be written as

$$\begin{aligned} \Pr[\mathbf{a}^\top \mathbf{x} \leq \ c] &\geq \alpha \\ \mathbf{x} &\in \{0,1\}^n \end{aligned}$$

- As $\mathbf{a}^\top \mathbf{x} \sim N(\bar{\mathbf{a}}^\top \mathbf{x}, \mathbf{x}^\top \Sigma \mathbf{x})$, we can reformulate the constraint as

$$\begin{aligned} \bar{\mathbf{a}}^\top \mathbf{x} + \Phi^{-1}(\alpha)\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} &\leq c \\ \mathbf{x} &\in \{0,1\}^n, \end{aligned}$$

where $\Phi(\cdot)$ represents the cumulative distribution function of the standard normal distribution

Chance-constrained knapsack inequality

- Given $\mathbf{a} \sim N(\bar{\mathbf{a}}, \Sigma)$ and $\alpha \in [0,1]$, a chance-constrained knapsack inequality can be written as

$$\Pr[\mathbf{a}^\top \mathbf{x} \leq c] \geq \alpha$$
$$\mathbf{x} \in \{0,1\}^n$$

- As $\mathbf{a}^\top \mathbf{x} \sim N(\bar{\mathbf{a}}^\top \mathbf{x}, \mathbf{x}^\top \Sigma \mathbf{x})$, we can reformulate the constraint as

$$\bar{\mathbf{a}}^\top \mathbf{x} + \Phi^{-1}(\alpha)\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} \leq c$$
$$\mathbf{x} \in \{0,1\}^n,$$

  where $\Phi(\cdot)$ represents the cumulative distribution function of the standard normal distribution
- Task: *Enumerate* all maximal feasible solutions for the inequality

Some interesting special cases

1. Fixed-rank case: covariance matrix Σ has *completely positive* (cp) *rank* d, i.e., we can find a matrix $A \in \mathbb{R}_+^{d\times n}$ such that $\Sigma = A^\top A$
   - Example: $a = A^\top z + \bar{a}$, where $z_1, \ldots, z_d \sim N(0,1)$ are i.i.d.'s.
   - We can rewrite the inequality as a *second order cone* inequality:

$$\|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq t \qquad \left(\text{where } \mathbf{b} = \frac{\bar{\mathbf{a}}}{\Phi^{-1}(\alpha)} \text{ and } t = \frac{c}{\Phi^{-1}(\alpha)}\right)$$

$$\mathbf{x} \in \{0,1\}^n$$

Some interesting special cases

1. Fixed-rank case: covariance matrix Σ has *completely positive* (cp) *rank* d, i.e., we can find a matrix $A \in \mathbb{R}_+^{d \times n}$ such that $\Sigma = A^\top A$
   - Example: $a = A^\top z + \bar{a}$, where $z_1, \ldots, z_d \sim N(0,1)$ are i.i.d.'s.
   - We can rewrite the inequality as a *second order cone* inequality:

$$\|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq t \qquad \left(\text{where } \mathbf{b} = \frac{\bar{\mathbf{a}}}{\Phi^{-1}(\alpha)} \text{ and } t = \frac{c}{\Phi^{-1}(\alpha)}\right)$$

$$\mathbf{x} \in \{0,1\}^n$$

2. Independent case: item sizes are independent: $A = D$ is a full-rank diagonal matrix
   - $a_j \sim N(\bar{a}_j, d_{jj})$ are independent

Some interesting special cases

1. Fixed-rank case: covariance matrix Σ has *completely positive* (cp) *rank d*, i.e., we can find a matrix $A \in \mathbb{R}_+^{d \times n}$ such that $\Sigma = A^\top A$
   - Example: $a = A^\top z + \bar{a}$, where $z_1, \ldots, z_d \sim N(0,1)$ are i.i.d.'s.
   - We can rewrite the inequality as a *second order cone* inequality:

$$\|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq t \qquad \left(\text{where } \mathbf{b} = \frac{\bar{\mathbf{a}}}{\Phi^{-1}(\alpha)} \text{ and } t = \frac{c}{\Phi^{-1}(\alpha)}\right)$$

$$\mathbf{x} \in \{0,1\}^n$$

2. Independent case: item sizes are independent: $A = D$ is a full-rank diagonal matrix
   - $a_j \sim N(\bar{a}_j, d_{jj})$ are independent
3. Simple independent case: $A = D$ is a full-rank diagonal matrix and there the elements of $[n]$ can be ordered s.t. $\bar{a}_1 \geq \cdots \geq \bar{a}_n$ and $d_{11} \geq \cdots \geq d_{nn}$
   - E.g.: $a_j \sim N(\bar{a}_j, 1)$ are independent

Monotone systems

- Consider a monotone system of inequalities of the form:

$$
\begin{aligned}
f_i(\mathbf{x}) &\leq t_i, \quad \text{for } i \in [r] := \{1, \ldots, r\} \\
\mathbf{x} &\in \{0,1\}^n
\end{aligned}
$$

- $f_i : \{0,1\}^n \mapsto \mathbb{R}_+$ is a *monotone* (non-decreasing) non-negative function on $\{0,1\}^n$:

$$\mathbf{x}, \mathbf{y} \in \{0,1\}^n \text{ and } \mathbf{x} \geq \mathbf{y} \quad \text{imply } f_i(\mathbf{x}) \geq f_i(\mathbf{y})$$

- *Maximal feasible* vector (solution): $\mathbf{x} \in \{0,1\}^n$ s.t. $\mathbf{x}$ is feasible for the system and $\mathbf{x} + \mathbf{1}^j$ is not feasible for all $j \in [n]$
- *Minimal infeasible* vector: $\mathbf{x} \in\in \{0,1\}^n$ is.t. $\mathbf{x}$ is infeasible for the system and $\mathbf{x} - \mathbf{1}^j$ is feasible for all $j \in [n]$ such that $x_j > 0$

Enumerating maximal feasible solutions for a monotone System

- Consider a monotone system of inequalities of the form:

$$\begin{aligned} f_i(\mathbf{x}) \leq \ & t_i, \quad \text{for } i \in [r] := \{1, \ldots, r\} \\ \mathbf{x} \in \{0,1\}^n & \end{aligned}$$

- $\mathcal{F}$: set of maximal feasible vectors
- $\mathcal{G}$: set of minimal infeasible vectors

Enumerating maximal feasible solutions for a monotone System

- Consider a monotone system of inequalities of the form:

$$\begin{aligned} f_i(\mathbf{x}) \leq \ & t_i, \quad \text{for } i \in [r] := \{1, \ldots, r\} \\ \mathbf{x} \in \{0,1\}^n & \end{aligned}$$

- $\mathcal{F}$: set of maximal feasible vectors
- $\mathcal{G}$: set of minimal infeasible vectors
- We are interested in incrementally generating the family $\mathcal{F}$:

  $\mathrm{GEN}(\mathcal{F}')$: *Given a monotone system, and a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of its maximal feasible vectors, either find a new maximal vector $\mathbf{x} \in \mathcal{F} \setminus \mathcal{F}'$, or state that $\mathcal{F}' = \mathcal{F}$.*

Enumerating maximal feasible solutions for a monotone System

- Consider a monotone system of inequalities of the form:

$$
\begin{aligned}
f_i(\mathbf{x}) \leq \; & t_i, \quad \text{for } i \in [r] := \{1, \ldots, r\} \\
\mathbf{x} \in \{0,1\}^n &
\end{aligned}
$$

- $\mathcal{F}$: set of maximal feasible vectors
- $\mathcal{G}$: set of minimal infeasible vectors
- We are interested in incrementally generating the family $\mathcal{F}$:

  $\text{GEN}(\mathcal{F}')$: *Given a monotone system, and a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of its maximal feasible vectors, either find a new maximal vector $\mathbf{x} \in \mathcal{F} \setminus \mathcal{F}'$, or state that $\mathcal{F}' = \mathcal{F}$.*

- Output-sensitive algorithm: running time depends on both input size n, *bit length* and output size $|\mathcal{F}|$

Monotone Boolean Dualization (MBD)

- Given a Boolean function f in CNF:

$$f(\mathbf{x}) = \bigwedge_{F\in\mathcal{F}} \bigvee_{i\in F} x_i$$

- The objective is to write f in *irredundant* DNF:

$$f(\mathbf{x}) = \bigvee_{G\in\mathcal{G}} \bigwedge_{i\in G} x_i$$

- Well-known problem with many applications
- Can be solved in time $poly(n) + m^{o(\log m)}$ time, where $m = |\mathcal{F}| + |\mathcal{G}|$ [Fredman and Khachiyan (1996)]
- Output-sensitive algorithm: running time depends on both input size $|\mathcal{F}| + n$ and output size $|\mathcal{G}|$

Monotone Boolean Dualization (MBD)

- Example: Input $\mathcal{F} = \big\{\{1,2,4\},\{2,3\},\{3,4\}\big\}$

$$\begin{aligned} f(\mathbf{x}) &= (x_1 \vee x_2 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \\ &= x_1x_3 \vee x_2x_3 \vee x_2x_4 \vee x_1x_2x_4 \vee x_1x_3x_4 \end{aligned}$$

- Same as finding *minimal feasible solutions* of the following systems:

$$\begin{aligned} x_1 + x_2 + x_4 &\geq 1 \\ x_2 + x_3 &\geq 1 \\ x_3 + x_4 &\geq 1 \\ \mathbf{x} \in \{0,1\}^n \end{aligned}$$

$$(x_1 + x_2 + x_4)(x_2 + x_3)(x_3 + x_4) \geq 1$$
$$\mathbf{x} \in \{0,1\}^n$$

$$1 - (1-x_1)(1-x_2)(1-x_4) + (1-x_2)(1-x_3) + (1-x_3)(1-x_4) \geq 1$$
$$\mathbf{x} \in \{0,1\}^n$$

Monotone Boolean Dualization (MBD)

- Example: Input $\mathcal{F} = \{\{1,2,4\},\{2,3\},\{3,4\}\}$

$$\begin{aligned} f(\mathbf{x}) &= (x_1 \vee x_2 \vee x_4) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \\ &= x_1x_3 \vee x_2x_3 \vee x_2x_4 \vee x_1x_2x_4 \vee x_1x_3x_4 \end{aligned}$$

- Same as finding *maximal* feasible solutions of the following systems:

$$\begin{aligned} x_1 + x_2 + x_4 &\leq 2 \\ x_2 + x_3 &\leq 1 \\ x_3 + x_4 &\leq 1 \\ \mathbf{x} \in \{0,1\}^n & \end{aligned}$$

$$12 - (3 - x_1 - x_2 - x_4)(2 - x_2 - x_3)(2 - x_3 - x_4) \leq 11$$
$$\mathbf{x} \in \{0,1\}^n$$

$$x_1x_2x_4 + x_2x_3 + x_3x_4 \leq 0, \quad \mathbf{x} \in \{0,1\}^n$$

Monotone systems

- Examples
  - *Linear* functions: $f_i(\mathbf{x}) := \sum_j a_{ij}x_j$, where $a_{ij} \geq 0$

- Examples
  - *Linear* functions: $f_i(\mathbf{x}) := \sum_j a_{ij} x_j$, where $a_{ij} \geq 0$
  - *Polynomials*:
    - $f_i(\mathbf{x}) = \sum_{H \in \mathcal{H}_i} a_H \prod_{j \in H} x_j$, where $\mathcal{H}_i \subseteq 2^{[n]}$ is a given mutliset family with $a_H \geq 0$ for all $H \in \mathcal{H}_i$
    - $f_i(\mathbf{x}) = R_i - \prod_k \left(\sum_i a_{ijk}(1 - x_j) \right)$, where $a_{ijk} \geq 0$, $R_i = \prod_i \left(\sum_j a_{ij} \right)$

- Examples
  - *Linear* functions: $f_i(\mathbf{x}) := \sum_j a_{ij} x_j$, where $a_{ij} \geq 0$
  - *Polynomials*:
    - $f_i(\mathbf{x}) = \sum_{H \in \mathcal{H}_i} a_H \prod_{j \in H} x_j$, where $\mathcal{H}_i \subseteq 2^{[n]}$ is a given mutliset family with $a_H \geq 0$ for all $H \in \mathcal{H}_i$
    - $f_i(\mathbf{x}) = R_i - \prod_k \left(\sum_i a_{ijk}(1 - x_j)\right)$, where $a_{ijk} \geq 0$, $R_i = \prod_i \left(\sum_j a_{ij}\right)$
  - *Supermodular* functions:

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}) \qquad \forall \mathbf{x}, \mathbf{y} \in \{0, 1\}$$

  where
    - $(\mathbf{x} \vee \mathbf{y})_j = \max\{x_j, y_j\}$
    - $(\mathbf{x} \wedge \mathbf{y})_j = \min\{x_j, y_j\}$

Known results

- If the f_i's are linear then the enumeration problem (of maximal feasible solutions of the system) is polynomially equivalent to MBD, and thus can be solved in quasi-polynomial time:
  - More precisely, problem $GEN(\mathcal{F}, \mathcal{F}')$ can be solved in *quasi-polynomial* time $k^{o(\log k)}$ time, where $k = \max\{n, r, |\mathcal{F}'|\}$ [Boros et al. (2000)]

Known results

- If the f_i's are linear then the enumeration problem (of maximal feasible solutions of the system) is polynomially equivalent to MBD, and thus can be solved in quasi-polynomial time:
  - More precisely, problem $GEN(\mathcal{F}, \mathcal{F}')$ can be solved in *quasi-polynomial* time $k^{o(\log k)}$ time, where $k = \max\{n, r, |\mathcal{F}'|\}$ [Boros et al. (2000)]
- If the f_i's are polynomials of the form $f_i(\mathbf{x}) = \sum_{H \in \mathcal{H}_i} a_H \prod_{j \in H} x_j$, then $GEN(\mathcal{F}, \mathcal{F}')$ is polynomially equivalent to MBD [Boros et al. (2004)]

Known results

- If the f_i's are linear then the enumeration problem (of maximal feasible solutions of the system) is polynomially equivalent to MBD, and thus can be solved in quasi-polynomial time:
  - More precisely, problem $GEN(\mathcal{F}, \mathcal{F}')$ can be solved in *quasi-polynomial* time $k^{o(\log k)}$ time, where $k = \max\{n, r, |\mathcal{F}'|\}$ [Boros et al. (2000)]
- If the f_i's are polynomials of the form $f_i(\mathbf{x}) = \sum_{H \in \mathcal{H}_i} a_H \prod_{j \in H} x_j$, then $GEN(\mathcal{F}, \mathcal{F}')$ is polynomially equivalent to MBD [Boros et al. (2004)]
- If the f_i's are supermodular with *integer* range $\{0, 1, \ldots, R\}$, then problem is quasi-polynomially equivalent to MBD provided that $R =$ quasi-poly(Input size):
  - More precisely, problem $GEN(\mathcal{F}, \mathcal{F}')$ can be solved in *quasi-polynomial* time $k^{o(\log k \cdot \log(R-t))}$ time, where $k = \max\{n, r, |\mathcal{F}'|\}$ [Boros et al. (2002)]

Chance constrained multi-dimensional knapsack inequality

- So we know how to "efficiently" enumerate maximal feasible solutions for a *multi-dimensional* knapsack problem:

$$\sum_j a_{ij} x_j \leq t_i, \quad \text{for } i \in [r] := \{1, \ldots, r\}$$
$$\mathbf{x} \in \{0,1\}^n$$

- For a single inequality ($r = 1$), enumeration can be done in polynomial time [Peled and Simeone (1985), Crama (1987)]
- What about the chance-constrained version?

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

- When $b = \mathbf{0}$, we can square to reduce to the polynomial case:

$$\mathbf{x}^\top A^\top A\mathbf{x} \leq t^2$$

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

- When $b = \mathbf{0}$, we can square to reduce to the polynomial case:

$$\mathbf{x}^\top A^\top A\mathbf{x} \leq t^2$$

- When $\mathbf{b} \neq \mathbf{0}$, squaring does not yield an equivalent problem, and may also result in a term with a *negative* coefficient:

$$\mathbf{x}^\top A^\top A\mathbf{x} + 2t\mathbf{b}^\top \mathbf{x} - (\mathbf{b}^\top \mathbf{x})^2 \leq t^2$$

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

- When $b = \mathbf{0}$, we can square to reduce to the polynomial case:

$$\mathbf{x}^\top A^\top A\mathbf{x} \leq t^2$$

- When $\mathbf{b} \neq \mathbf{0}$, squaring does not yield an equivalent problem, and may also result in a term with a *negative* coefficient:

$$\mathbf{x}^\top A^\top A\mathbf{x} + 2t\mathbf{b}^\top \mathbf{x} - (\mathbf{b}^\top \mathbf{x})^2 \leq t^2$$

- Example 1: $\sqrt{x_1 + x_2} + 2x_1 \leq 1 \rightarrow$ Squaring: $x_1 + x_2 \leq 1$
- Example 2: $\sqrt{x_1 + x_2} + x_1 + x_2 \leq 2 \rightarrow$ Squaring: $2x_2 + 2x_2 - x_1x_2 \leq 2$

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

- Is $f(\mathbf{x})$ supermodular?

Fact

A function $f : \{0,1\} \to \mathbb{R}_+$ is supermodular if and only if, for any $j \in [n]$, and for any $\mathbf{x} \in \{0,1\}^n$ s.t. $x_j = 0$, the difference

$$\partial_f(\mathbf{x}, j) := f(\mathbf{x} + \mathbf{1}^j) - f(\mathbf{x}),$$

is monotone (increasing) in $\mathbf{x}$

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

- Is $f(\mathbf{x})$ supermodular?

Fact

A function $f : \{0,1\} \to \mathbb{R}_+$ is supermodular if and only if, for any $j \in [n]$, and for any $\mathbf{x} \in \{0,1\}^n$ s.t. $x_j = 0$, the difference

$$\partial_f(\mathbf{x}, j) := f(\mathbf{x} + \mathbf{1}^j) - f(\mathbf{x}),$$

is monotone (increasing) in $\mathbf{x}$

- Example: $\|D\mathbf{x}\| + \mathbf{b}^T\mathbf{x}$ for a diagonal matrix D
- $\partial_f(\mathbf{x}, j) = \frac{d_{jj}^2}{\|D(\mathbf{x}+\mathbf{1}^j)\| + \|D\mathbf{x}\|} + b_j$ is *decreasing* in $\mathbf{x}$ $\Rightarrow$ f is *submodular*

Second-order cone inequality

- Consider a second-order cone inequality:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \; t$$

where $A \in \mathbb{R}_+^{d\times n}$ and $\mathbf{b} \in \mathbb{R}_+^n$ are given matrix and vector

Theorem

If $d = O(1)$, then all maximal feasible vectors for the second-order cone inequality can be enumerated in quasi-polynomial time

Enumeration via joint-generation

- Proof is via reduction to MBD via *dual-boundedness*:
  - $\mathcal{F}$: set of maximal feasible vectors
  - $\mathcal{G}$: set of minimal infeasible vectors
  - We know that MBD is equivalent to enumerating $\mathcal{F} \cup \mathcal{G}$ [Bioch and T. Ibaraki (1995), Gurvich and Khachiyan (1999)]
  - Only need to show that $|\mathcal{G}|$ is "small"

Enumeration via joint-generation

- Proof is via reduction to MBD via *dual-boundedness*:
  - $\mathcal{F}$: set of maximal feasible vectors
  - $\mathcal{G}$: set of minimal infeasible vectors
  - We know that MBD is equivalent to enumerating $\mathcal{F} \cup \mathcal{G}$ [Bioch and T. Ibaraki (1995), Gurvich and Khachiyan (1999)]
  - Only need to show that $|\mathcal{G}|$ is "small"
  - We show that $|\mathcal{G}| = O(n)^{2d+1}|\mathcal{F}|$

Simple independent case

- Consider the simple independent case (item sizes are independent and $b_1 \geq \cdots \geq b_n$ and $d_{11} \geq \cdots \geq d_{nn}$)
- Then, $f(\mathbf{x}) = \sqrt{\sum_j d_{jj}^2 x_j} + \mathbf{b}^\top \mathbf{x}$
- Use *2-monotonicity* [Crama (1987)]:

$$k < i \Rightarrow f(\mathbf{y} - \mathbf{1}^i + \mathbf{1}^k) - f(\mathbf{y}) \geq 0$$

- This gives $|\mathcal{G}| \leq n|\mathcal{F}|$

Simple independent case

- Consider the simple independent case (item sizes are independent and $b_1 \geq \cdots \geq b_n$ and $d_{11} \geq \cdots \geq d_{nn}$)
- Then, $f(\mathbf{x}) = \sqrt{\sum_j d_{jj}^2 x_j} + \mathbf{b}^\top \mathbf{x}$
- Use *2-monotonicity* [Crama (1987)]:

$$k < i \Rightarrow f(\mathbf{y} - \mathbf{1}^i + \mathbf{1}^k) - f(\mathbf{y}) \geq 0$$

  - We have

$$f(\mathbf{y} - \mathbf{1}^i + \mathbf{1}^k) - f(\mathbf{y}) = \frac{d_{kk}^2 - d_{ii}^2}{\sqrt{\sum_{j \neq i,k} d_{jj}^2 y_j + d_{kk}^2} + \sqrt{\sum_{j \neq i,k} d_{jj}^2 y_j + d_{ii}^2}} + b_k - b_i \geq 0$$

- This gives $|\mathcal{G}| \leq n|\mathcal{F}|$

Simple independent case

- Consider the simple independent case (item sizes are independent and $b_1 \geq \cdots \geq b_n$ and $d_{11} \geq \cdots \geq d_{nn}$)
- Then, $f(\mathbf{x}) = \sqrt{\sum_j d_{jj}^2 x_j} + \mathbf{b}^\top \mathbf{x}$
- Use *2-monotonicity* [Crama (1987)]:

$$k < i \Rightarrow f(\mathbf{y} - \mathbf{1}^i + \mathbf{1}^k) - f(\mathbf{y}) \geq 0$$

  - We have

$$f(\mathbf{y} - \mathbf{1}^i + \mathbf{1}^k) - f(\mathbf{y}) = \frac{d_{kk}^2 - d_{ii}^2}{\sqrt{\sum_{j \neq i,k} d_{jj}^2 y_j + d_{kk}^2} + \sqrt{\sum_{j \neq i,k} d_{jj}^2 y_j + d_{ii}^2}} + b_k - b_i \geq 0$$

- This gives $|\mathcal{G}| \leq n|\mathcal{F}|$
- Argument does not work if $A = D \neq I$ is a (general) diagonal matrix

Proof of dual-boundedness

- First consider a linear system:

$$\sum_i a_{ij} x_j \leq \; t_i, \qquad \text{for } i \in [r] := \{1, \ldots, r\}$$
$$\mathbf{x} \in \{0,1\}^n$$

- Each vector $\mathbf{a}^i := (a_{ij} \mid j \in [n])$ defines (at least) one permutation $\sigma_{\mathbf{a}^i}$ given by: $a_{i,\sigma(1)} \geq a_{i,\sigma(2)} \geq \cdots \geq a_{i,\sigma(n)}$
- It is known [Crama (1987), Boros et al. (2000)] that

$$|\mathcal{G}| \leq r' \cdot n|\mathcal{F}|$$

- $r' :=$ number of distinct permutations defined by the set of vectors $\mathbf{a}^1, \ldots, \mathbf{a}^r$

Proof of dual-boundedness

- Consider the SOC inequaltiy:

$$f(\mathbf{x}) := \|A\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \leq \ t$$

- Let us rewrite the SOC inequality as

$$f_{\mathbf{u}}(\mathbf{x}) := \mathbf{u}^\top A\mathbf{x} + \mathbf{b}^\top \mathbf{x} \leq \ t, \qquad \text{for } \mathbf{u} \in \mathbb{B}_+^d(0,1)$$

where $\mathbb{B}_+^d(0,1) := \{\mathbf{x} \in \mathbb{R}_+^d : \ \|\mathbf{x}\| \leq 1\}$ the non-negative half of the d-dimensional unit ball centered at the origin

- This is a semi-infinite LP: the **u**-th constraint is defined by the weight vector: $\mathbf{w}^{\mathbf{u}} := A^\top \mathbf{u} + \mathbf{b} \in \mathbb{R}_+^n$
- The question reduces to: How many distinct permutations defined by the set of weights $\{\mathbf{w}^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{B}_+^d(0,1)\}$?

Number of distinct permutations

Fact

Any arrangement of m d-dimensional hyperplanes partitions $\mathbb{R}^d$ into at most $\Phi_d(m) := \sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{em}{d}\right)^d$ maximal connected regions not intersected by any of the hyperplanes (called cells of the arrangement).

Number of distinct permutations

Fact

Any arrangement of m d-dimensional hyperplanes partitions $\mathbb{R}^d$ into at most $\Phi_d(m) := \sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{em}{d}\right)^d$ maximal connected regions not intersected by any of the hyperplanes (called cells of the arrangement).

- We show that

$$|\{\sigma_{\mathbf{a}^{\mathbf{u}}} \mid \mathbf{u} \in \mathbb{B}_+^d(0,1)\}| \leq \Phi_d(n(n-1)/2) = O(n)^{2d}$$

Number of distinct permutations

Fact

Any arrangement of m d-dimensional hyperplanes partitions $\mathbb{R}^d$ into at most $\Phi_d(m) := \sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{em}{d}\right)^d$ maximal connected regions not intersected by any of the hyperplanes (called cells of the arrangement).

- We show that

$$|\{\sigma_{\mathbf{a}^{\mathbf{u}}} \mid \mathbf{u} \in \mathbb{B}_+^d(0,1)\}| \leq \Phi_d(n(n-1)/2) = O(n)^{2d}$$

  - Write $A = [\mathbf{a}^1, \ldots, \mathbf{a}^n]$ where $\mathbf{a}^j \in \mathbb{R}_+^d$ is the j-th column of A
  - Then $w_j = w_j^{\mathbf{u}} = (\mathbf{a}^j)^\top \mathbf{u} + b_j$

Number of distinct permutations

Fact

Any arrangement of m d-dimensional hyperplanes partitions $\mathbb{R}^d$ into at most $\Phi_d(m) := \sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{em}{d}\right)^d$ maximal connected regions not intersected by any of the hyperplanes (called cells of the arrangement).

- We show that

$$|\{\sigma_{\mathbf{a}^{\mathbf{u}}} \mid \mathbf{u} \in \mathbb{B}^d_+(0,1)\}| \leq \Phi_d(n(n-1)/2) = O(n)^{2d}$$

  - Write $A = [\mathbf{a}^1, \ldots, \mathbf{a}^n]$ where $\mathbf{a}^j \in \mathbb{R}^d_+$ is the j-th column of A
  - Then $w_j = w_j^{\mathbf{u}} = (\mathbf{a}^j)^\top \mathbf{u} + b_j$
  - The system of inequalities $w_j \leq w_{j'}$ for $j, j' \in [n]$ (considering $\mathbf{u}$ as a variable in $\mathbb{R}^d$) defines a hyperplane arrangement:

$$\left(\mathbf{a}^j - \mathbf{a}^{j'}\right)^\top \mathbf{u} \leq b_{j'} - b_j, \quad \text{for } j \neq j' \in [n].$$

Chance-constraint multi-dimensional knapsack

- Given vectors $\mathbf{a}^i = (a_{ij} \mid j \in [n])$ drawn from a *multivariate normal distribution* with mean $\bar{\mathbf{a}}^i \in \mathbb{R}^n_+$ and covariance matrix $\Sigma^i \succeq 0$, i.e., $\mathbf{a}^i \sim N(\bar{\mathbf{a}}^i, \Sigma^i)$:

$$\Pr[(\mathbf{a}^i)^\top \mathbf{x} \leq \ t_i] \geq \alpha_i, \qquad \text{for } i \in [r],$$
$$\mathbf{x} \in \{0,1\}^n$$

- We can show that $|\mathcal{G}| = O(n)^{2d+1} r|\mathcal{F}|$, where $d := \max_i d_i$
- Consequently, if $d = O(1)$, then all maximal feasible vectors can be enumerated in quasi-polynomial time

Enumerating minimal feasible solutions

- Consider the *covering* inequality

$$g(\mathbf{x}) := \|D\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \geq t$$
$$\mathbf{x} \in \{0,1\}^n$$

  where $D \in \mathbb{R}_+^{n\times n}$ is a diagonal matrix (independent case)
- $\mathcal{F}$: set of maximal *infeasible* vectors
- $\mathcal{G}$: set of minimal *feasible* vectors
- Previous argument does not work
  - and even if it works, it would give $|\mathcal{F}| = O(n)^{2n+1}|\mathcal{G}|$

Enumerating minimal feasible solutions

- Consider the *covering* inequality

$$g(\mathbf{x}) := \|D\mathbf{x}\| + \mathbf{b}^\top \mathbf{x} \geq t$$
$$\mathbf{x} \in \{0,1\}^n$$

where $D \in \mathbb{R}_+^{n \times n}$ is a diagonal matrix (independent case)

- $\mathcal{F}$: set of maximal *infeasible* vectors
- $\mathcal{G}$: set of minimal *feasible* vectors
- Previous argument does not work
  - and even if it works, it would give $|\mathcal{F}| = O(n)^{2n+1}|\mathcal{G}|$
- Better bound via supermodularity: $|\mathcal{F}| \leq |\mathcal{G}|^{o(\log \frac{t}{\tau})}$:

Enumerating minimal feasible solutions

- Better bound via supermodularity: $|\mathcal{F}| \leq |\mathcal{G}|^{o(\log \frac{t}{\tau})}$:
    - $f(\mathbf{x}) := R - \|D(\mathbf{1} - \mathbf{x})\| - \mathbf{b}^\top(\mathbf{1} - \mathbf{x}) \leq R - t$

Enumerating minimal feasible solutions

- Better bound via supermodularity: $|\mathcal{F}| \le |\mathcal{G}|^{o(\log \frac{t}{\tau})}$:
  - $f(\mathbf{x}) := R - \|D(\mathbf{1}-\mathbf{x})\| - \mathbf{b}^\top(\mathbf{1}-\mathbf{x}) \le R - t$
  - *Range*: $R := \sqrt{n}D_{\max} + nb_{\max}$, where $D_{\max} := \max_j d_{jj}$ and $b_{\max} := \max_j b_j$
  - *Traction*:

$$\tau := \min_{\substack{j\in[n],\ \mathbf{x}\in\{0,1\},\ x_j=0 \\ f(\mathbf{x}+\mathbf{1}^j)>f(\mathbf{x})}} f(\mathbf{x}+\mathbf{1}^j) - f(\mathbf{x}) \ge \min\left\{\frac{D_{\min}^2}{2\sqrt{n}D_{\max}}, b_{\min}\right\}$$

where $D_{\min} = \min\{d_{jj} \mid d_{jj} > 0\}$, $b_{\min} = \min\{b_j \mid b_j > 0\}$

Enumerating minimal feasible solutions

- Better bound via supermodularity: $|\mathcal{F}| \leq |\mathcal{G}|^{o(\log \frac{t}{\tau})}$:
  - $f(\mathbf{x}) := R - \|D(\mathbf{1}-\mathbf{x})\| - \mathbf{b}^\top(\mathbf{1}-\mathbf{x}) \leq R - t$
  - *Range*: $R := \sqrt{n}D_{\max} + nb_{\max}$, where $D_{\max} := \max_j d_{jj}$ and $b_{\max} := \max_j b_j$
  - *Traction*:

$$\tau := \min_{\substack{j\in[n],\ \mathbf{x}\in\{0,1\},\ x_j=0\\ f(\mathbf{x}+\mathbf{1}^j)>f(\mathbf{x})}} f(\mathbf{x}+\mathbf{1}^j) - f(\mathbf{x}) \geq \min\left\{\frac{D_{\min}^2}{2\sqrt{n}D_{\max}}, b_{\min}\right\}$$

where $D_{\min} = \min\{d_{jj} \mid d_{jj} > 0\}$, $b_{\min} = \min\{b_j \mid b_j > 0\}$

- Based on extending the result in [Boros et al. (2002)] for *integer-valued* supermodular functions
- Consequently, if the entries of D and b are polynomial in n, we get a quasi-polynomial time enumeration algorithm

Some open questions

- Can we show $|\mathcal{G}| = \text{poly}(n, d, |\mathcal{F}|)$ (for the SOC inequality)?
  - in comparison to $|\mathcal{G}| = O(n)^{2d+1}|\mathcal{F}|$
- Is there a polynomial time enumeration algorithm for a single chance-constraint knapsack inequality?
  - in comparison to a quasi-polynomial time algorithm

Thank you