Matroid Horn functions

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Joint work with **Endre Boros** and **Kazuhisa Makino**

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Boolean Seminar Liblice
September 26, 2023

Long story short...

A long time ago in a city far, far away...
(May 2018, Budapest)

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Let's associate Horn functions to matroids!

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Good idea, there are a lot of nice questions here!

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Sure, but we could also consider hypergraphs.

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New observations on implicate duality!

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Great, we should wrap up everything. 

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Outline

Hypergraph Horn functions

Hypergraphs

Given a finite set V, a hypergraph is a family $\mathcal{H} \subseteq 2^V$. The hypergraph is Sperner if $H_1 \not\subset H_2$ for any $H_1, H_2 \in \mathcal{H}$.

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Remark: $(\mathcal{H}^c)^c = \mathcal{H}$.

$T \subseteq V$ is a transversal of $\mathcal{H}$ if $T \cap H \neq \emptyset$ for every $H \in \mathcal{H}$. The family of minimal transversals of $\mathcal{H}$ is denoted by $\mathcal{H}^d$.

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The intersection closure of $\mathcal{H}$ is

$$\mathcal{H}^\cap = \left\{ \bigcap_{F \in \mathcal{F}} F \;\middle|\; \mathcal{F} \subseteq \mathcal{H} \right\}.$$

True sets, keys, and closure

For a definite Horn function $h\colon 2^V \to \{0,1\}$, $\mathcal{T}(h)$ is the family of true sets of h. For $Z \subseteq V$, $\mathbb{T}_h(Z)$ is the unique minimal true set containing Z. The set is closed if $\mathbb{T}_h(Z) = Z$.

Remark: $\mathbb{T}_h(Z)$ is the so-called forward-chaining closure of Z.

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The family of maximal nontrivial true sets of h is

$$\mathcal{M}(h) = \{T \subsetneq V \mid h(T) = 1 \text{ and } h(T') = 0 \text{ for all } T \subsetneq T' \subsetneq V\}.$$

Circular CNFs

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A definite Horn function $h\colon 2^V \to \{0,1\}$ is hypergraph Horn if $h = \Phi_{\mathcal{H}}$ for some hypergraph $\mathcal{H} \subseteq 2^V$.

Remarks:

- $\Phi_{\mathcal{H}} = \Phi_{\mathcal{H}'}$ might hold even if $\mathcal{H}$ is Sperner while $\mathcal{H}'$ is not.
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$\Rightarrow$ *Stay tuned for Endre's talk!*

Matroids

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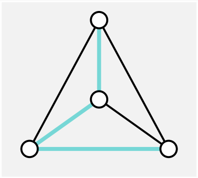
Given a ground set V, a matroid $\mathbb{M} = (V, \mathcal{I})$ is a pair where $\mathcal{I} \subseteq 2^V$ satisfies the independence axioms:

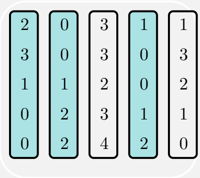
(I1) $\emptyset \in \mathcal{I}$,
(I2) $X \subseteq Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$,
(I3) $X, Y \in \mathcal{I}$, $|X| < |Y| \Rightarrow \exists e \in X - Y$ s.t. $X + e \in \mathcal{I}$.

Introduced by Hassler Whitney (1935) and by Takeo Nakasawa (1935).

Examples:



Graphic matroid



Linear matroid

Circuits, closed sets, hyperplanes

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(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$, then $C_1 \not\subset C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct and $u \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - u$.

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A hyperplane is a closed set of rank $r(V) - 1$.

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Q1: Characterization of matroid Horn functions?
Q2: Connection between Boolean and matroid terminology?
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For a matroid Horn function h, a CNF representation $h = \Phi_{\mathcal{C}}$ is canonical if $\mathcal{C}$ satisfies the circuit axioms (C1)–(C3).

For a Boolean function $f : 2^V \to \{0,1\}$, the unique CNF representation that contains all prime implicates of f is the complete CNF of f.

Boolean vs. matroid terminology

Lemma

Let $\mathcal{C}$ be the family of circuits of a matroid $\mathbb{M}$, and let $h = \Phi_{\mathcal{C}}$. Then we have $\mathbb{T}_h(X) = \mathrm{cl}_{\mathbb{M}}(X)$ for all $X \subseteq V$.

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Matroid $\mathbb{M}$ with circuit family $\mathcal{C}$	**Matroid Horn function h represented by $\Phi_{\mathcal{C}}$**
Bases of $\mathbb{M}$	Minimal keys of h
Closed sets of $\mathbb{M}$	True sets of h
Hyperplanes of $\mathbb{M}$	Maximal nontrivial true sets of h

Characterizations

Characterizations in terms of canonical representations.

B, Boros, Makino, '23

Let $\mathcal{C} \subseteq 2^V$ and let $h = \Phi_{\mathcal{C}}$. Then the following are equivalent.

(i) $\mathcal{C}$ satisfies circuit axiom (C3).

(ii) $\mathcal{K}(h) = \mathcal{C}^{dc}$.

(ii) $\mathcal{M}(h) = \mathcal{C}^{dcdc}$.

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Characterizations in terms of complete CNF.

B, Boros, Makino, '23

For a definite Horn function h, the following are equivalent.

(i) The function h is matroid Horn.
(ii) The complete CNF of h is circular.
(iii) The implicate-dual function h^i is matroid Horn.
(iv) The complete CNF of h^i is circular.

Minimum representations

Objectives

Given $\mathcal{F} \subseteq 2^V$ and $\mathcal{G} \subseteq \mathcal{F}$, let

$\langle \mathcal{G} \rangle^1_{\mathcal{F}} = \mathcal{G} \cup \{X \in \mathcal{F} \mid X = (X_1 \cup X_2) - v \text{ for distinct } X_1, X_2 \in \mathcal{G},\ v \in X_1 \cap X_2\}.$

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Remarks:

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- If $\langle \mathcal{G} \rangle_{\mathcal{F}}^k = \langle \mathcal{G} \rangle_{\mathcal{F}}^{k+1}$, then the final system is denoted by $\langle \mathcal{G} \rangle_{\mathcal{F}}$, and $\mathcal{G}$ is a generator of $\mathcal{F}$ if $\langle \mathcal{G} \rangle_{\mathcal{F}} = \mathcal{F}$.

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For a matroid $\mathbb{M} = (V, \mathcal{C})$, let $h_{\mathbb{M}}$ be the corresponding matroid Horn function.
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(G) circuit generator: $|\mathbb{M}|_G =$ min cardinality of a generator of $\mathcal{C}$,

[$\approx$ Complexity of $\mathcal{C}$.]

(C) $\sharp$ of circuits: $|\mathbb{M}|_C =$ min cardinality of a subsystem $\mathcal{D} \subseteq \mathcal{C}$ s.t. $h_{\mathbb{M}} = \Phi_{\mathcal{D}}$,

(K) $\sharp$ of circuit clauses: $|\mathbb{M}|_K =$ min $\sharp$ of circuit clauses needed to represent $h_{\mathbb{M}}$.

[$\approx$ Minimum number of clasues, but restricted to circuit clauses.]

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Lemma

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$\Rightarrow$ $C \in \mathcal{C}$ is chordless if there is no $C' \in \mathcal{C}$ with $|C' \setminus C| = 1$ and $|C'| < |C|$.

Binary matroids

Recall: A matroid is binary if it is a linear matroid over $\mathbb{Z}_2$.

Lemma

Let $\mathbb{M} = (V, \mathcal{C})$ be a simple binary matroid and $X \subseteq V$ be an independent set. Then there is at most one $v \in V$ for which $X + v$ forms a circuit of $\mathbb{M}$.

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Let $\mathbb{M} = (V, \mathcal{C})$ be a simple binary matroid. Then the set of chordless cycles is the unique optimal solution with respect to $|\mathbb{M}|_G$, $|\mathbb{M}|_C$ and $|\mathbb{M}|_K$.

Uniform matroids

Recall: A matroid is uniform if $\mathcal{C} = \{C \subseteq V \mid |C| = r + 1\}$.

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Let $\mathbb{M} = (V, \mathcal{C})$ be a rank-r uniform matroid on $n \geq r+2$ elements. Then $\binom{n}{r}/(r+\frac{1}{2}) \leq |\mathbb{M}|_C \leq \binom{n}{r}$.

Sketch of the proof.

Upper bound: For an arbitrary $v \in V$, let $\mathcal{D} = \{X + v \mid X \subseteq V - v, |X| = r\}$. Then $F_{\Phi_\mathcal{D}}(X) = X$ if $|X| \leq r-1$ and $F_{\Phi_\mathcal{D}}(X) = V$ otherwise.

Lower bound: If every r-element set X shares 1 token evenly among the sets in $\mathcal{D}$ containing it, then every set in $\mathcal{D}$ receives at most $r + \frac{1}{2}$ tokens in total.

Turán systems

An $(r+1)$-uniform hypergraph $\mathcal{H} \subseteq 2^V$ is

- a covering $(n, r+1, r)$-system if every $X \subseteq V$ of size r is contained in at least one hyperedge (min size: $c(n, r+1, r)$),
- a Steiner $(n, r+1, r)$-system if every $X \subseteq V$ of size r is contained in exactly one hyperedge (min size: $s(n, r+1, r)$),
- an implication $(n, r+1, r)$-system if for every $X \subseteq V$ of size at least r, there exists a hyperedge $H \in \mathcal{H}$ with $|H \setminus X| = 1$ (min size: $b(n, r+1, r)$).

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Horn-logic interpretation: $\mathcal{H} \subseteq 2^V$ is

- a covering $(n, r+1, r)$-system if $\mathbb{T}_{\Phi_{\mathcal{H}}}(X) \neq X$ for all $X \subseteq V, |X| = r$,
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$\Rightarrow$ For a rank-r uniform matroid, $\mathcal{D} \subseteq \mathcal{C}$ satisfy $h_{\mathbb{M}} = \Phi_{\mathcal{D}}$ if and only if $\mathcal{D}$ forms an implication $(n, r+1, r)$-system.

Lower and upper bounds

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Let $\mathbb{M} = (V, \mathcal{C})$ be a rank-r uniform matroid on $n \geq r + 1$ elements. Then

$$c(n, r+1, r) \leq |\mathbb{M}|_C \leq 2 \cdot c(n, r+1, r).$$

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Let $\mathbb{M} = (V, \mathcal{C})$ be a rank-2 uniform matroid on $n \geq 46$ elements. Then

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Let $\mathbb{M} = (V, \mathcal{C})$ be a rank-2 uniform matroid on $n \geq 46$ elements. Then

$$|\mathbb{M}|_C = \frac{n^2}{5} + O(n).$$

Open problem: Given a rank-r uniform matroid $\mathbb{M}$, what is the right order of magnitude of $|\mathbb{M}|_C = b(n, r+1, r)$?

Open problem: What is the computational complexity of checking if a given a definite Horn function h represented by a definite Horn CNF Ψ is matroid Horn or not?

Thank you for your attention...

K. Bérczi, E. Boros, M. Kazuhisa, Matroid Horn functions, arXiv:2301.06642 (2023).

K. Bérczi, E. Boros, M. Kazuhisa, Hypergraph Horn functions, arXiv:2301.05461 (2023).

...and happy birthday, Endre!