Decomposing 1-Sperner Boolean functions, with applications to graphs

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Endre Boros[1], Vladimir Gurvich[1,2], Martin Milanič[3]

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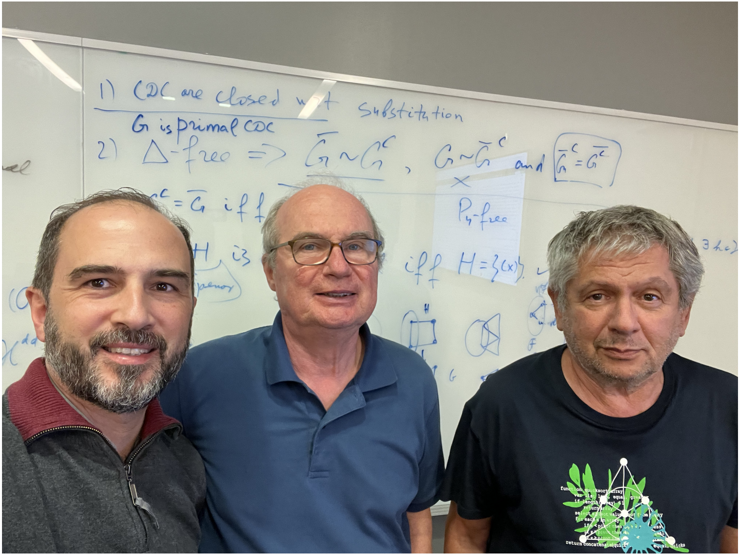
[1] Rutgers University, New Jersey, USA

[2] National Research University Higher School of Economics, Moscow, Russia

[3] University of Primorska, Koper, Slovenia

Happy Birthday Endre!!



Goal of the talk

To describe:

- a class of combinatorially defined threshold monotone Boolean functions,
- a **construction** of all such functions,
- **applications** to structural and algorithmic graph theory.

Given two binary vectors $x, y \in \{0,1\}^n$, we write $x \leq y$ if $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$.

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is **monotone** if for all $x, y \in \{0,1\}^n$:

$$x \leq y \quad \Rightarrow \quad f(x) \leq f(y)\,.$$

A Boolean function is monotone if and only if it can be represented by a monotone DNF.

Example:

$$f(x_1, x_2, x_3, x_4) = x_1 x_2 \vee x_3 x_4 \vee x_1 x_2 x_4$$

An **implicant** of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is a conjuction C of literals that implies f:

$$C(x) \leq f(x) \text{ for all } x \in \{0,1\}^n .$$

An implicant is **prime** if it does not imply any other implicant.

Example: For the function f given by the DNF

$$x_1x_2 \vee x_3x_4 \vee x_1x_2x_4 ,$$

each of the three terms of the DNF is an implicant of f.
However, the implicant $x_1x_2x_4$ is not prime, since it implies the implicant x_1x_2.

Every monotone Boolean function has a **unique irredundant DNF**, namely the disjunction of all its prime implicants.

Example: The unique irredundant DNF of the function f from the previous example is $x_1x_2 \vee x_3x_4$.

Every Boolean function $f : \{0,1\}^n \to \{0,1\}$ partitions the Boolean hypercube $\{0,1\}^n$ into two sets V_0 and V_1, where

$V_0 = \{x \in \{0,1\}^n : f(x) = 0\}$ is the set of **false points** of f, and

$V_1 = \{x \in \{0,1\}^n : f(x) = 1\}$ is the set of **true points** of f.

A Boolean function f is **threshold** if its sets of false points and true points can be separated by a hyperplane, that is, if there exist real numbers $w_1, \ldots, w_n, t$ such that

$$f(x) = 0 \quad \text{if and only if} \quad \sum_{i=1}^{n} w_i x_i \leq t\,.$$

Question: Given a monotone Boolean function represented by a DNF, how difficult it is to check if it is threshold?

Theorem (Peled, Simeone, 1985)

There is a polynomial-time algorithm that determines, given a monotone Boolean function f represented by a DNF, whether f is threshold.

The algorithm is based on linear programming.

Recall that the goal of the talk is to describe:

- a class of combinatorially defined threshold monotone Boolean functions,
- a **construction** of all such functions,
- **applications** to structural and algorithmic graph theory.

A **hypergraph** is a pair (V, E) where V is a set of **vertices** and E is a set of subsets of V called **hyperedges**.

To every monotone Boolean function $f : \{0,1\}^n \to \{0,1\}$ we associate its **prime implicant hypergraph** $\mathcal{H}_f$:

- $V = \{1, \ldots, n\}$,
- a set $S \subseteq \{1, \ldots, n\}$ is a hyperedge if and only if $\bigwedge_{i \in S} x_i$ is a prime implicant of f.

By construction, the prime implicant hypergraph $\mathcal{H}_f$ is always **Sperner**: no hyperedge contains another.

Example:

Let $f = x_1 x_2 \vee x_3 x_4$.

Then $\mathcal{H}_f = (\{1,2,3,4\}, \{\{1,2\}, \{3,4\}\})$.

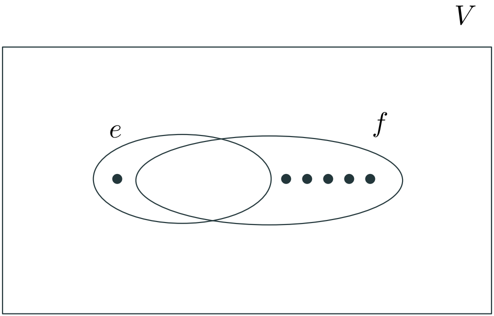
Note that a hypergraph $\mathcal{H} = (V, E)$ is Sperner if and only if any two distinct hyperedges $e, f \in E$ satisfy

$$\min\{|e \setminus f|, |f \setminus e|\} \geq 1\,.$$

A hypergraph is $\mathcal{H} = (V, E)$ 1-Sperner if any two distinct hyperedges $e, f \in E$ satisfy

$$\min\{|e \setminus f|, |f \setminus e|\} = 1\,,$$

that is, if **for every two hyperedges** the smallest of the two set differences is of size one.



By definition, a monotone Boolean function f is **1-Sperner** if its prime implicant hypergraph is 1-Sperner.

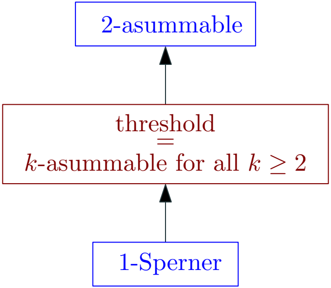
Proposition (Chiarelli, M., WG 2013, DAM 2014)

Every 1-Sperner Boolean function is threshold.

The proof does not show how to compute the threshold weights.

It is based on a characterization of threshold Boolean functions via asummability due to Chow (1961) and Elgot (1961)
(which itself relies on Farkas' Lemma / separation of convex polyhedra).

In general, the geometrically defined thresholdness property is sandwiched between two combinatorially defined properties:



A Boolean function f is **2-summable** if there exist two true points x^1, x^2 and two false points y^1, y^2 such that

$$x^1 + x^2 = y^1 + y^2 .$$

It is **2-asummable** if it is not 2-summable.

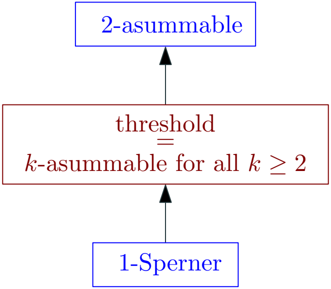
Example: The function $f = x_1 x_2 \vee x_3 x_4$ is 2-summable, as verified by:

$x^1 = (1, 1, 0, 0)$, $x^2 = (0, 0, 1, 1)$

$y^1 = (1, 0, 1, 0)$, $y^2 = (0, 1, 0, 1)$

In particular, f is not threshold, since it is not 2-asummable.

In general, none of the implications in the diagram can be reversed.



However, for some classes of monotone Boolean functions arising from graphs, 2-asummability implies thresholdness or even 1-Spernerness.

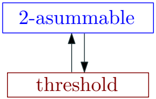
In such cases, the thresholdness property admits a simple combinatorial characterization.

To every graph G, we can associate a hypergraph $\mathcal{H}$ with vertex set $V(G)$ in many ways, for example, by taking for hyperedges:

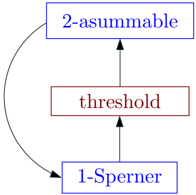
- the edges,
- the minimal vertex covers,
- the maximal cliques,
- the maximal independent sets,
- the minimal closed neighborhoods,
- the minimal dominating sets.

Let f be a monotone Boolean function such that its prime implicant hypergraph $\mathcal{H}_f$ is of one of the above types.

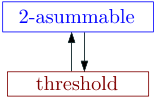
When is f threshold?



- the edges
- the minimal vertex covers,
- the maximal cliques
- the maximal independent sets
- the minimal closed neighborhoods

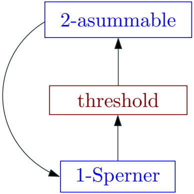


- the minimal dominating sets



threshold graphs
(Chvátal, Hammer, 1977)

- the edges
- the minimal vertex covers,
- the maximal cliques
- the maximal independent sets
- the minimal closed neighborhoods
- the minimal dominating sets



domishold graphs
(Benzaken, Hammer, 1978)

Reference: Boros, Gurvich, M. Characterizing and decomposing classes of threshold, split, and bipartite graphs via 1-Sperner hypergraphs, *J. Graph Theory* 94 (2020) 364–397.

Construction of 1-Sperner hypergraphs

$\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ – vertex-disjoint hypergraphs

z – a new vertex

The **gluing of** $\mathcal{H}_1$ **and** $\mathcal{H}_2$ is the hypergraph

$$\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$$

such that

$$V(\mathcal{H}) = V_1 \cup V_2 \cup \{z\}$$

and

$$E(\mathcal{H}) = \{\{z\} \cup e \mid e \in E_1\} \cup \{V_1 \cup e \mid e \in E_2\}.$$

The operation of gluing can be visualized easily in terms of **incidence matrices**.

Every hypergraph $\mathcal{H} = (V, E)$ with

$V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$

can be represented with its

incidence matrix $A^{\mathcal{H}} \in \{0,1\}^{m \times n}$:

rows are indexed by hyperedges of $\mathcal{H}$,

columns are indexed by vertices of $\mathcal{H}$,

and

$$A^{\mathcal{H}}_{i,j} = \begin{cases} 1, & \text{if } v_j \in e_i; \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$, then

$$A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{pmatrix} \mathbf{1}^{m_1,1} & A^{\mathcal{H}_1} & \mathbf{0}^{m_1,n_2} \\ \mathbf{0}^{m_2,1} & \mathbf{1}^{m_2,n_1} & A^{\mathcal{H}_2} \end{pmatrix}.$$

Example:

$$A^{\mathcal{H}_1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A^{\mathcal{H}_2} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow \quad A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{pmatrix} z & & & & & & & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Proposition

For every pair $\mathcal{H}_1 = (V_1, E_1)$ *and* $\mathcal{H}_2 = (V_2, E_2)$ *of vertex-disjoint* 1*-Sperner hypergraphs,*

their gluing $\mathcal{H}_1 \odot \mathcal{H}_2$ *is a* 1*-Sperner hypergraph,*

unless $E_1 = \{V_1\}$ *and* $E_2 = \{\emptyset\}$

(in which case $\mathcal{H}_1 \odot \mathcal{H}_2$ *is not Sperner).*

$$A^{\mathcal{H}_1} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$A^{\mathcal{H}_2} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \quad A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{pmatrix} \overset{z}{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We show that every 1-Sperner hypergraph that has a vertex can be generated as a gluing of two smaller 1-Sperner hypergraphs.

We say that a gluing of two vertex-disjoint hypergraphs $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ is **safe** unless $E_1 = \{V_1\}$ and $E_2 = \{\emptyset\}$.

Theorem

A hypergraph $\mathcal{H}$ is 1-Sperner if and only if

it either has no vertices (that is, $\mathcal{H} \in \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\})\}$)

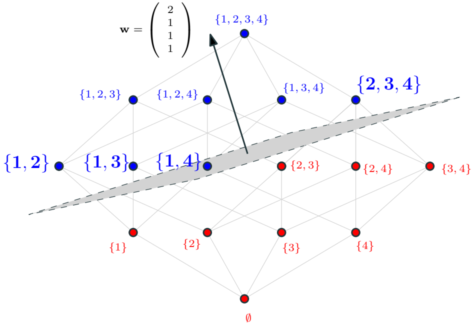
or it is a safe gluing of two smaller 1-Sperner hypergraphs.

Reference: Boros, Gurvich, M. Decomposing 1-Sperner hypergraphs
The Electronic Journal of Combinatorics 26(3) (2019), #P3.18.

Consequences

The construction for 1-Sperner hypergraphs has several **consequences**:

1. An alternative proof of the fact that **every 1-Sperner Boolean function is threshold**.

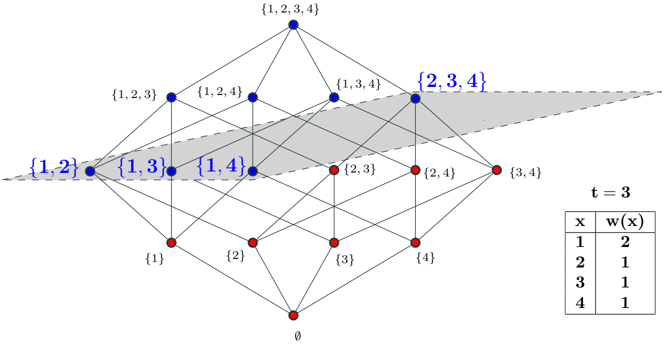


$\mathbf{t = 3}$

x	**w(x)**
1	**2**
2	**1**
3	**1**
4	**1**

The proof is constructive and builds the corresponding weights and threshold.

2. A proof of the fact that every 1-Sperner hypergraph is **equilizable**.



$\mathbf{t = 3}$

x	w(x)
1	2
2	1
3	1
4	1

Equilizable hypergraphs can be seen as a generalization of **equistable graphs** (introduced in 1980 by Payan and studied afterwards in many papers).

3. An upper bound on the size of 1-Sperner hypergraphs.

Proposition

For every 1*-Sperner hypergraph* $\mathcal{H} = (V, E)$ *with* $E \neq \{\emptyset\}$*, we have* $|E| \leq |V|$*.*

- Proof idea: we show that the characteristic vectors of the hyperedges are linearly independent over the reals.

4. A lower bound on the size of certain 1-Sperner hypergraphs.

Proposition

For every 1-Sperner hypergraph $\mathcal{H} = (V, E)$ with $|V| \geq 2$ and without universal, isolated, and twin vertices, we have

$$|E| \geq \left\lceil \frac{|V| + 2}{2} \right\rceil .$$

This bound is sharp.

Given a hypergraph $\mathcal{H} = (V, E)$, a **transversal** of $\mathcal{H}$ is a set of vertices intersecting all hyperedges of $\mathcal{H}$.

The **transversal hypergraph** $\mathcal{H}^T$ is the hypergraph with vertex set V in which a set $S \subseteq V$ is a hyperedge if and only if S is an inclusion-minimal transversal of $\mathcal{H}$.

5. An efficient algorithm for computing the transversal hypergraph of a given 1-Sperner hypergraph.

Theorem

The number of minimal transversals of every 1*-Sperner hypergraph* $\mathcal{H} = (V, E)$ *is at most*

$$\max\left\{1, |V|, \binom{|V|}{2}\right\}.$$

This bound is sharp.

Moreover, the transversal hypergraph $\mathcal{H}^T$ *of a given* 1*-Sperner hypergraph* $\mathcal{H} = (V, E)$ *can be computed in time* $\mathcal{O}(|V|^3|E|)$.

Applications to graphs: new classes of bounded clique-width

The construction of 1-Sperner hypergraphs leads to

new classes of bipartite, cobipartite, and split graphs of bounded clique-width.

clique-width of a graph $G = (V, E)$ = smallest number of labels in a k-expression constructing (a graph isomorphic to) G

A k-**expression** is an algebraic expression building a graph together with labels $\ell(v) \in \{1, \ldots, k\}$ for all $v \in V$,

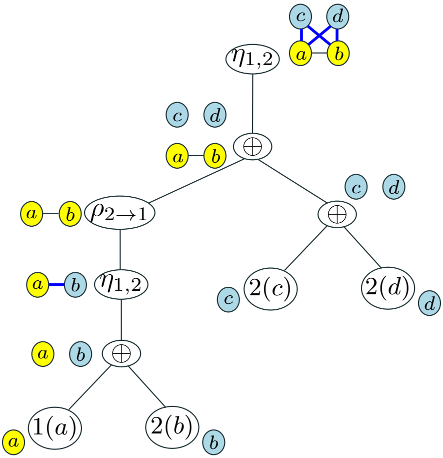
using the following operations:

- ℓ_v, $\ell \in \{1, \ldots, k\}$: creating a new one-vertex graph with vertex v labeled ℓ,
- $G \oplus H$: disjoint union
- $\eta_{i,j}(G)$ for $i, j \in \{1, \ldots, k\}, i \neq j$: add edges
- $\rho_{i \to j}(G)$ for $i, j \in \{1, \ldots, k\}, i \neq j$: relabel

Example:

The following expression builds a complete graph minus an edge using only labels 1 and 2 (yellow and blue, respectively):

$$\eta_{1,2}\left(\rho_{2\to 1}\left(\eta_{1,2}\left(1(a)\oplus 2(b)\right)\right)\oplus\left(2(c)\oplus 2(d)\right)\right)$$

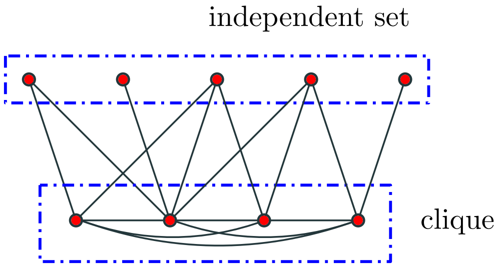


Many algorithmic decision or optimization problems on graphs that are NP-hard for general graphs can be solved **in polynomial time on graph classes of bounded clique-width** (often in linear time if a k-expression is known).

In particular, there is a metatheorem of Courcelle, Makowsky, and Rotics (2000).

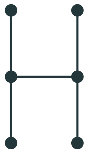
We illustrate the idea on the class of **split graphs** (introduced by Földes and Hammer in 1977).

A graph is **split** if has a **split partition**, that is, a partition (C, I) of the vertex set into a clique and an independent set.



The clique-width of split graphs is known to be unbounded.

It remains unbounded even for H-free split graphs, as shown by Brandstädt, Dabrowski, Huang, Paulusma (in 2016).



H

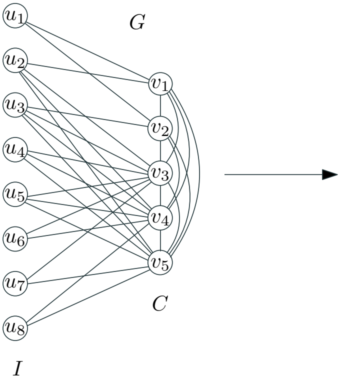
We show that the construction of 1-Sperner hypergraphs leads to

a 5-expression of a given H-free split graph G with a split partition (C, I) provided that

the neighborhoods in I of vertices in C are pairwise incomparable.

Idea of the transformation

Given a split graph G with a split partition (C, I), define a hypergraph $\mathcal{H} = (V, E)$ with $V = I$ and $E = \{N(v) \cap I : v \in C\}$.

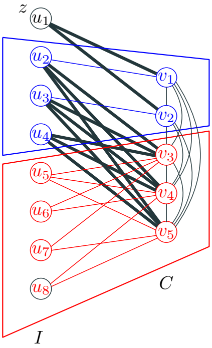


$$A^{\mathcal{H}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Since the neighborhoods in I of vertices in C are pairwise incomparable, $\mathcal{H}$ is **Sperner**.

Since G is H-free, $\mathcal{H}$ is also 1-**Sperner**.

We decompose …



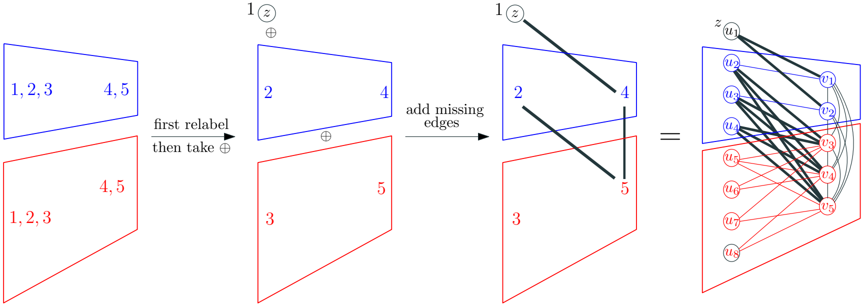
$$A^{\mathcal{H}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} \overset{z}{\overset{\parallel}{u_1}} & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

incidence matrix of hypergraph $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$

$$A^{\mathcal{H}_1} = \begin{array}{c} \\ v_1 \\ v_2 \end{array} \begin{pmatrix} u_2 & u_3 & u_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^{\mathcal{H}_2} = \begin{array}{c} \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

... and use induction!



This result leads to **efficient algorithms** for three basic variants of the **dominating set problem** in the class of H-free split graphs.

Given a graph $G = (V, E)$, a set $S \subseteq V$ is:

- a **dominating set** if every vertex in $V \setminus S$ has a neighbor in S,
- a **total dominating set** if every vertex has a neighbor in S,
- a **connected dominating set** if it is a dominating set that induces a connected subgraph of G.

Theorem

The problems of finding a minimum dominating set / total dominating set / connected dominating set

are solvable in time $\mathcal{O}(|V(G)|^3)$ in the class of H-free split graphs.

This result is sharp in the sense that all three problems are known to be NP-hard:

- in the class of split graphs,
- in the class of H-free graphs.

Thank you!

Questions?