Hypergraph Horn Functions

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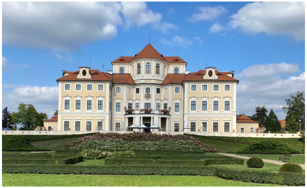

*Based on joint work with K. Bérczi and K. Makino

Boolean functions, clauses, and implicates, ...

Set of Variables: V, $|V| = n > 1$.

Set of Literals: $\mathbb{L} = \{v, \bar{v} = 1 - v \mid v \in V\}$.

Boolean Functions: $f : 2^V \mapsto \{0, 1\}$.

True Sets: $\mathcal{T}(f) = \{S \subseteq V \mid f(S) = 1\}$.

Clauses: A *clause* is a disjunction of literals:

$$\bar{1} \vee 2 \vee 3 \vee \bar{4}.$$

CNFs: A *CNF* is a conjunction of clauses.

Implicates: A clause C is an *implicate* of a Boolean function f if it evaluates to true whenever f does:

$$f \leq C.$$

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Definite Horn Functions

Definite (pure) Horn clause: Has exactly one unnegated literal: for $v \in V$, $B \subseteq V$

$$\left(v \vee \bigvee_{u \in B} \bar{u} \right) \quad \Longleftrightarrow \quad B \rightarrow v$$

Definite Horn Function h: Can be represented by a CNF in which every clause is definite Horn $\Longleftrightarrow$ $\mathcal{T}(h)$ is closed under intersections and $V \in \mathcal{T}(h)$ (Horn, 1951).

Implicates of a definite Horn function h:

$$h \quad \leq \quad B \xrightarrow{h} v$$

for some $B \subseteq V$ and $v \in V$.

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Implicate Sets

Implicate set: Given a Boolean function $f : 2^V \mapsto \{0,1\}$ a subset $I \subseteq V$ is an *implicate set* of f if

$$I - v \xrightarrow{f} v \quad \forall v \in I.$$

Equivalence: If $I \subseteq V$, $|I| = 2$ is an implicate set, then we have equivalent variables.

A (healthy) redundancy in databases: ...

Given $f : 2^V \mapsto \{0,1\}$ we denote by $\mathcal{I}(f)$ the family of its implicate sets.

Never empty: $\emptyset \in \mathcal{I}(f)$.

Closed under union: $\mathcal{I}(f) = \mathcal{I}(f)^{\cup}$.

We call a clause $A \rightarrow v$ a *circular implicate* of f if $A + v \in \mathcal{I}(f)$.

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Hypergraph Horn Functions

Given a hypergraph $\mathcal{H} \subseteq 2^V$ we associate to it a definite Horn CNF

$$\Phi_{\mathcal{H}} = \bigwedge_{H \in \mathcal{H}} \left(\bigwedge_{v \in H} ((H - v) \to v) \right).$$

A Boolean function $f : 2^V \mapsto \{0, 1\}$ is called *hypergraph Horn* if there exists a hypergraph $\mathcal{H} \subseteq 2^V$ such that

$$f \sim \Phi_{\mathcal{H}}.$$

$\Longrightarrow \mathcal{H} \subseteq \mathcal{I}(f).$

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Implicate Duality

Given a Boolean function $f : 2^V \mapsto \{0,1\}$ we define its *implicate dual* f^i by

$$\mathcal{T}(f^i) \;=\; (\mathcal{I}(f))^c \;=\; \{V \setminus I \mid I \in \mathcal{I}(f)\}$$

Recall that $\mathcal{I}(f)$ is union closed and $\emptyset \in \mathcal{I}(f)$. Consequently, $\mathcal{I}(f)^c$ is intersection closed and $V \in \mathcal{I}(f)^c$.

Claim

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Some examples

Hypergraph Horn functions may have non-circular implicates: For instance, if $\mathcal{H} = \{123, 234\}$, then

$$12 \overset{\Phi_{\mathcal{H}}}{\longrightarrow} 4 \quad \text{but} \quad 14 \overset{\Phi_{\mathcal{H}}}{\not\longrightarrow} 2$$

A hypergraph Horn function may be represented by different hypergraphs:

$$\Phi_{12,23,34} \sim \Phi_{13,14,24}$$

The same example also shows taking unions of a representation does not yield all implicate sets, in general. In particular, not all union closed families can appear as $\mathcal{I}(h)$ of a definite Horn function h.

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Operators

Assume that h is a definite Horn function represented by a definite Horn CNF Ψ and $S \subseteq V$.

We denote by $\mathbb{T}_h(S) = \mathbb{T}_\Psi(S)$ the unique minimal true set $T \in \mathcal{T}(h)$ such that $S \subseteq T$.

We denote by $\mathbb{I}_h(S) = \mathbb{I}_\Psi(S)$ the unique maximal implicate set $I \in \mathcal{I}(h)$ such that $I \subseteq S$.

$\mathbb{T}_h(S)$ is known as the *forward-chaining* closure of S. It can be computed in polynomial time in the size of Ψ.

We call $\mathbb{I}_h(S)$ the *h-core* of S. **It can be also computed in polynomial time in the size of Ψ.**

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Duality Between True Sets and Implicate Sets

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For a hypergraph Horn function h we have that

$$\mathcal{T}(h) = \{T \subseteq V \mid \nexists I \in \mathcal{I}(h) \text{ with } |I \setminus T| = 1\} \quad (1)$$

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Characterization of Hypergraph Horn Functions

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For a definite Horn function h the following claims are equivalent:

(i) *h is hypergraph Horn.*

(ii) $h^{ii} = h$.

(iii) $\forall \;\; F \in 2^V \setminus \mathcal{T}(h) \quad \exists \;\; I \in \mathcal{I}(h)$ *such that* $|I \setminus F| = 1$.

(iv) $\forall \;\; F \in 2^V \setminus \mathcal{T}(h) \quad \exists \;\; u \in \mathbb{T}_h(F) \setminus F$ *such that* $v \in \mathbb{T}_h(F - v + u)$ *holds for all* $v \in F$ *with* $h(F - v) = 1$.

- (ii) shows that implication duality is an involution of the family of hypergraph Horn functions, and it generalizes matroid duality.
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(iii) $\forall \; F \in 2^V \setminus \mathcal{T}(h) \quad \exists \; I \in \mathcal{I}(h)$ *such that* $|I \setminus F| = 1$.

(iv) $\forall \; F \in 2^V \setminus \mathcal{T}(h) \quad \exists \; u \in \mathbb{T}_h(F) \setminus F$ *such that* $v \in \mathbb{T}_h(F - v + u)$ *holds for all* $v \in F$ *with* $h(F - v) = 1$.

- (ii) shows that implication duality is an involution of the family of hypergraph Horn functions, and it generalizes matroid duality.
- (iii) is a reformulation of (1).
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Characterization of Hypergraph Horn Functions

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For a definite Horn function h the following claims are equivalent:

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Implicate Sets

Assume that h is represented by a definite Horn CNF Ψ.

Lemma

Given $X, Y \subseteq V$ we can find $I \in \mathcal{I}(h)$ such that $X \subseteq I$, $I \cap Y = \emptyset$, or prove that no such implicate set of h exists in polynomial time in the size of Ψ.

Corollary

- *We can generate $\mathcal{I}(h)$ with polynomial delay (in the size of Ψ).*
- *Given $\mathcal{H} \subseteq 2^V$, we can decide if $\mathcal{H} = \mathcal{I}(h)$ for a Horn function h, in polynomial time in the size of $\mathcal{H}$.*

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Hypergraph Horn Recognition

Assume that h is represented by a definite Horn CNF Ψ.

Theorem

We can decide in polynomial time in the size of Ψ if $h \sim \Phi_{\mathcal{H}}$ for a hypergraph $\mathcal{H} \subseteq 2^V$, and if yes, output such a hypergraph with $|\mathcal{H}| \leq |V| \cdot \|\Psi\|$.

- The produced hypergraph may not be Sperner ...
- ...even if $h \sim \Phi_{\mathcal{S}}$ for some Sperner hypergraph $\mathcal{S}$.
- **Recognizing the existence of a Sperner hypergraph $\mathcal{H}$ with $\Psi \sim \Phi_{\mathcal{H}}$ is an open problem.**

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Implicate Duality

Theorem

Given definite Horn CNFs Ψ and Γ, we can decide $\Psi \geq \Gamma^i$ in $O(|V|^2 \cdot |\Psi| \cdot \|\Gamma\|)$ time.

- **Deciding $\Psi \leq \Gamma^i$ is an open problem, even if both CNFs are hypergraph Horn.**
- Deciding $\Psi = \Psi^i$ belongs to co-NP.

Theorem

For a definite Horn function h we have $h = h^i$ if and only if $\mathcal{H} = \mathcal{I}(h)$ is a maximal family with respect to the property of

$$|H \cap H'| \neq 1 \quad \forall\ H, H' \in \mathcal{H}.$$

- Note that we have $\mathcal{H} = \mathcal{H}^d$ if and only if $\mathcal{H}$ is a maximal **Sperner** family with respect to the property of

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Keys of Definite Horn Functions

A subset $K \subseteq V$ is a *key* of a definite Horn function h if $\mathbb{T}_h(K) = V$. We denote by $\mathcal{K}(h)$ the family of minimal keys of h. Note that we have $\mathcal{M}(h) = \mathcal{K}(h)^{dc}$, where $\mathcal{M}(h)$ denotes the family of maximal non-trivial true sets of h.

Claim

Given a Sperner hypergraph $\mathcal{K} \subseteq 2^V$, we have $\mathcal{K} = \mathcal{K}(h)$ for a definite Horn function h if and only if

(i) $\mathcal{K}^{dc} \subseteq \mathcal{T}(h)$, *and*

(ii) $\mathcal{K}^{+} \setminus \{V\} \subseteq 2^V \setminus \mathcal{T}(h)$.

Let us call a subset $I \subseteq V$ a *potential implicate set* for $\mathcal{K}$ if $\mathcal{K}^{dc} \subseteq \mathcal{T}(\Phi_{\{I\}})$. We denote by $\mathcal{P}(\mathcal{K})$ the family of potential implicate sets for $\mathcal{K}$.

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Note that $\emptyset \in \mathcal{P}(\mathcal{K})$ and that $\mathcal{P}(\mathcal{K})$ is union closed, by the definition of potential implicate sets. Thus, for any subset $S \subseteq V$ we have a unique maximal potential implicate set in S.

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For a Sperner hypergraph $\mathcal{K} \subseteq 2^V$ and subset $S \subseteq V$ the unique maximal potential implicate set within S can be computed in $O(|S| \cdot \|\mathcal{K}\|^2)$ time.

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Keys of Definite Horn Functions Cont'd

Theorem

For a Sperner hypergraph $\mathcal{K} \subseteq 2^V$ *we can decide in* $O(|V|^3 \cdot |\mathcal{K}| \cdot \|\mathcal{K}\|^2)$ *time if* $\mathcal{K} = \mathcal{K}(\Phi_{\mathcal{H}})$ *for a hypergraph* $\mathcal{H}$*, and if yes, we can construct such a hypergraph with* $|\mathcal{H}| \leq |V| \cdot |\mathcal{K}|$*.*