Generating maximal irredundant and minimal redundant subfamilies of a given hypergraph.

Endre Boros*

MSIS Department and RUTCOR, Rutgers University

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*Joint results with Kaz Makino (Kyoto University)

Basic Definitions and Questions

- $\mathcal{H} \subseteq 2^V$ is **irredundant** if every hyperedge of $\mathcal{H}$ has its **private** element:
$$\forall H \in \mathcal{H} \;\; \exists v \in H \quad \text{with} \quad \deg_{\mathcal{H}}(v) = 1$$
- Otherwise, it is called **redundant**.
- Being redundant is a monotone property over the subfamilies of 2^V, and being irredundant is its negation.
- We denote by $MinRED(\mathcal{H})$ the collection of minimal redundant subfamilies of $\mathcal{H}$.
- We denote by $MaxIRR(\mathcal{H})$ the collection of maximal irredundant subfamilies of $\mathcal{H}$.
- **Given $\mathcal{H} \subseteq 2^V$, what is the complexity of generating $MinRED(\mathcal{H})$ and/or $MaxIRR(\mathcal{H})$?**

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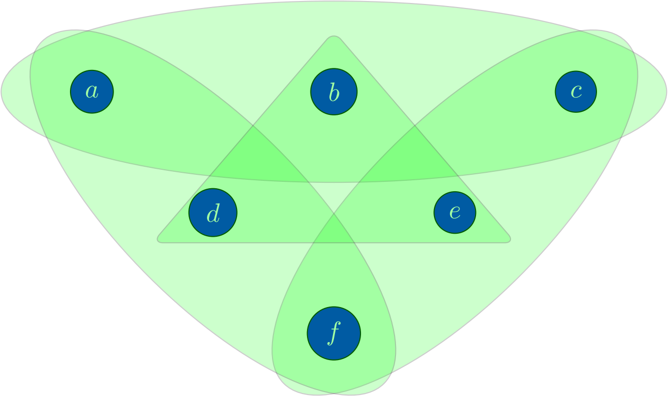

A minimal redundant hypergraph

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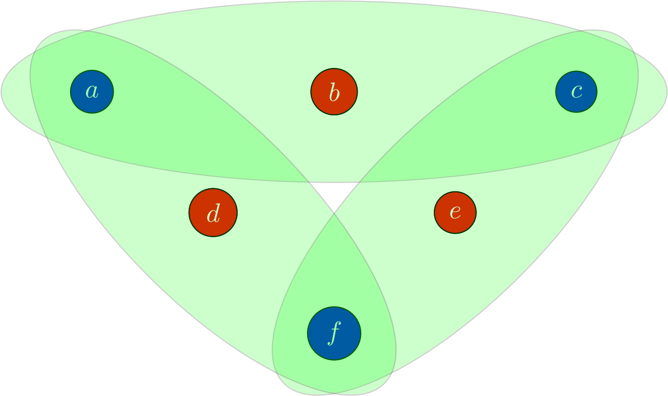

A maximal irredundant subhypergraph

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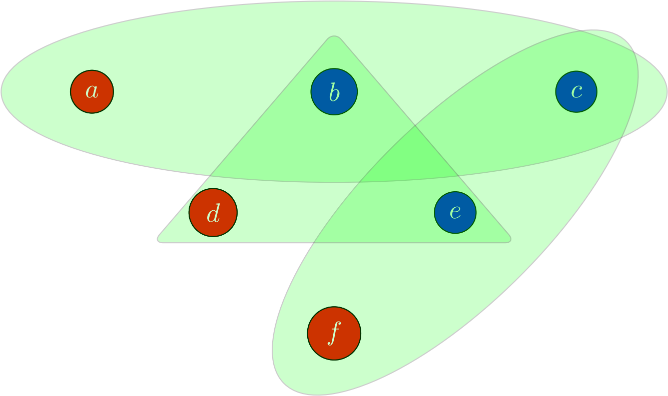

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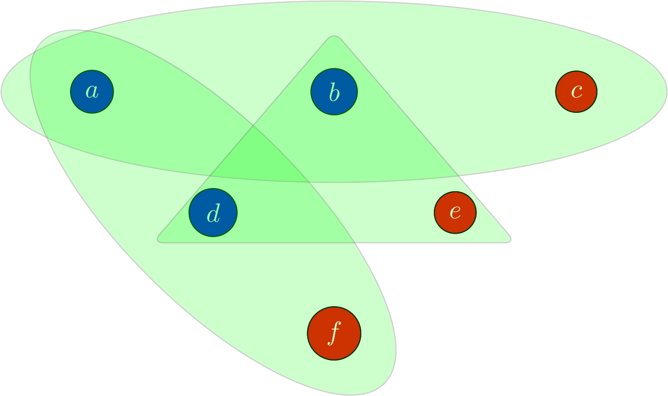

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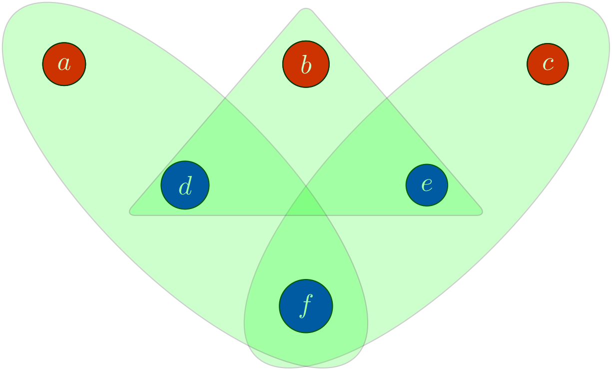

A maximal irredundant subhypergraph

Incidence Matrix Example – MaxIRR

	H_1	H_2	H_3	H_4	H_5
v_1	●			●	●
v_2	●				●
v_3		●		●	●
v_4		●		●	
v_5			●		
v_6	●	●			
v_7		●	●	●	●
v_8					●

Incidence Matrix Example – MaxIRR

	H_1	H_2	H_3	H_4	H_5
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v_2	●				●
v_3		●		●	●
v_4		●		●	
v_5			●		
v_6	●	●			
v_7		●	●	●	●
v_8					●

$\mathcal{H} = \{H_1, H_2, H_3, H_4, H_5\}$

$\{H_1, H_2, H_3\} \in MaxIRR(\mathcal{H})$

Incidence Matrix Example – MaxIRR

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v_1	●			●	●
v_2	●				●
v_3		●		●	●
v_4		●		●	
v_5			●		
v_6	●	●			
v_7		●	●	●	●
v_8					●

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Transversals and Dualization

- Given a hypergraph $\mathcal{H} \subseteq 2^V$ a subset $T \subseteq V$ is a **transversal** of $\mathcal{H}$ if
$$\forall H \in \mathcal{H} \quad \text{we have} \quad T \cap H \neq \emptyset.$$
- Denote by $MinTR(\mathcal{H})$ the set of all minimal transversals of $\mathcal{H}$.
- For a Sperner hypergraph $\mathcal{H}$ we have
$$MinTR(MinTR(\mathcal{H})) = \mathcal{H}$$
- **For $\mathcal{T} \subseteq MinTR(\mathcal{H})$ deciding $\mathcal{T} = MinTR(\mathcal{H})$ is solvable in quasi-polynomial time (Fredman and Khachiyan, 1996)**
- **Equivalent with monotone Boolean dualization ...**
- **Given $\mathcal{H} \subseteq 2^V$ generating $MinTR(\mathcal{H})$ is solvable in incremental quasi-polynomial time (Fredman and Khachiyan, 1996)**

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Minimal Covers

- Given a hypergraph $\mathcal{H} \subseteq 2^V$ we denote by $U(\mathcal{H})$ the union of its hyperedges
$$U(\mathcal{H}) \;=\; \{v \in V \mid \deg_{\mathcal{H}}(v) \geq 1\}$$
- $\mathcal{S} \subseteq \mathcal{H}$ is a **cover** of $U(\mathcal{H})$ if $U(\mathcal{S}) = U(\mathcal{H})$.
- Denote by $MinCOV(\mathcal{H})$ the collection of subfamilies of $\mathcal{H}$ that are minimal covers of $U(\mathcal{H})$.
- Consider $v = \{H \in \mathcal{H} \mid H \ni v\} \subseteq \mathcal{H}$. Then we can view V as $\subseteq 2^{\mathcal{H}}$. Sometimes V is called the transposed hypergraph of $\mathcal{H}$: $V = \mathcal{H}^T$.
- $MinTR(V) \;\longleftrightarrow\; MinCOV(\mathcal{H})$.
- **The problem of generating $MinCOV(\mathcal{H})$ for a given hypergraph $\mathcal{H}$ is another equivalent problem with monotone Boolean dualization.**

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v_2	●				●
v_3		●		●	●
v_4		●		●	
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$$MinCOV(\mathcal{H}) \updownarrow MinTR(\mathcal{H}^T)$$

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$$MinCOV(\mathcal{H})$$
$$\updownarrow$$
$$MinTR(\mathcal{H}^T)$$

Some Simple Connections and Motivation

- $MaxIRR(\mathcal{H})$ and $MinRED(\mathcal{H})$ are the maximal false and minimal true points of a monotone Boolean function over $\mathcal{H}$.
- **MinCOV**$(\mathcal{H}) \subseteq$ **MaxIRR**$(\mathcal{H})$.
- Irredundant sets appear in the process of generating minimal dominating sets in graphs (see more later.)
- $MinRED$ is not easier than monotone Boolean dualization (Takeaki Uno, 2015)
- **How hard are problems $MinRED$ and $MaxIRR$?**
- **Is there a polynomial relation between $|MinRED(\mathcal{H})|$ and $|MaxIRR(\mathcal{H})|$?**
- **What about special classes of hypergraphs?**

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- **Is there a polynomial relation between $|MinRED(\mathcal{H})|$ and $|MaxIRR(\mathcal{H})|$?**
- **What about special classes of hypergraphs?**

Some Simple Connections and Motivation

- $MaxIRR(\mathcal{H})$ and $MinRED(\mathcal{H})$ are the maximal false and minimal true points of a monotone Boolean function over $\mathcal{H}$.
- $\mathbf{MinCOV}(\mathcal{H}) \subseteq \mathbf{MaxIRR}(\mathcal{H})$.
- Irredundant sets appear in the process of generating minimal dominating sets in graphs (see more later.)
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A Weak but Educational Attempt for MaxIRR

- Given $\mathcal{H} \subseteq 2^V$ associate a graph $G_{\mathcal{H}} = (W, E)$ to $\mathcal{H}$, where

$$\begin{aligned} W &= \{(v, H) \mid v \in H \in \mathcal{H}\} \\ E &= \{((v, H), (v', H')) \mid v = v' \text{ or } H = H' \text{ or } v \in H' \text{ or } v' \in H\} \end{aligned}$$

- If $S = \{(v_i, H_i) \in W \mid i = 1, ..., k\} \subseteq W$ is a maximal independent set in $G_{\mathcal{H}}$, then $\mathcal{S} = \{H_i \mid i = 1, ..., k\} \in MaxIRR(\mathcal{H})$.
- If $\mathcal{S} = \{H_i \mid i = 1, ..., k\} \in MaxIRR(\mathcal{H})$ and $v_i \in H_i$ is the private element of H_i with respect to $\mathcal{S}$ for all $i = 1, ..., k$, then $S = \{(v_i, H_i) \mid i = 1, ..., k\} \subseteq W$ is a maximal independent set in $G_{\mathcal{H}}$.
- Maximal independent sets of a graph can be generated with polynomial delay (Tsukiyama, Ide, Ariyoshi, and Shirakawa, 1977; Johnson, Yannakakis and Papadimitriou, 1980).
- **Does not imply that $MaxIRR(\mathcal{H})$ can be generated with polynomial delay! Exponential overcount!!**

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- Let $A \in \{0, 1\}^{V \times \mathcal{H}}$ be the incidence matrix of $\mathcal{H}$.
- Maximal subsets of columns in A that include a unit submatrix of the same size correspond to $MaxIRR(\mathcal{H})$ in a one-to-one way.
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Incidence Matrix Example – MaxIRR

	H_1	H_2	H_3	H_4	H_5
v_1	●			●	●
v_2	●				●
v_3		●		●	●
v_4		●		●	
v_5			●		
v_6	●	●			
v_7		●	●	●	●
v_8					●

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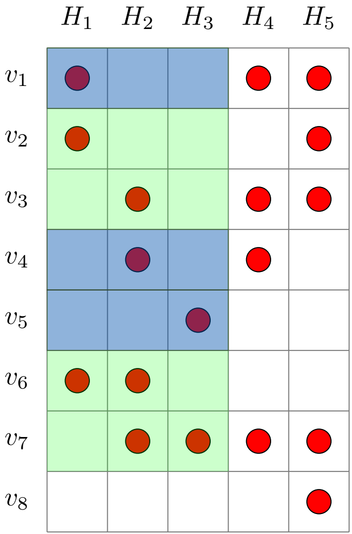

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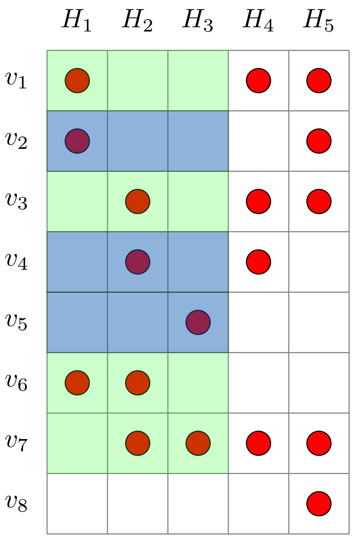

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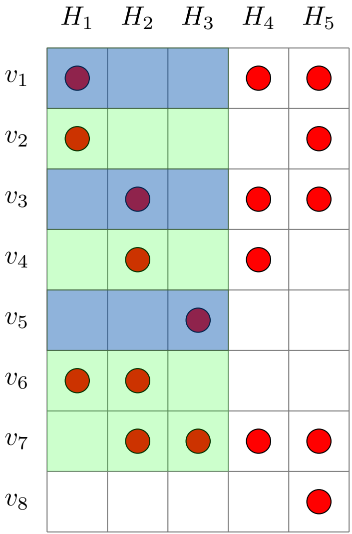

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Main Results

- Given a hypergraph $\mathcal{H} \subseteq 2^V$, we denote by

$$\Delta(\mathcal{H}) = \max_{v \in V} \deg_{\mathcal{H}}(v)$$

its **maximum degree**.

Theorem 1

Problem MinRED is co-NP-complete, even if it is restricted to hypergraphs with $\Delta(\mathcal{H}) = 3$.

Theorem 2

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Main Results Cont'd

- Given a graph $G = (V, E)$ a set $D \subseteq V$ is **dominating** if for all $v \in V$ we have $N_G[v] \cap D \neq \emptyset$. We denote by $MinDOM(G)$ the collection of all minimal dominating sets of G.
- Denoting by $\mathcal{N}_G = \{N_G[v] \mid v \in V\}$ we have

$$MinDOM(G) = MinTR(\mathcal{N}_G)$$

- Problem $MinDOM$ is polynomially equivalent with monotone Boolean dualization (Kant, Limouzy, Mary, and Nourine, 2014).

Theorem 3

When the input is restricted to hypergraphs $\mathcal{H}$ with $\Delta(\mathcal{H}) = 2$ then problem $MinRED$ is trivial, while problem $MaxIRR$ is polynomially equivalent with $MinDOM$ over all graphs.

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If the input is restricted to hypergraphs $\mathcal{H}$ with $\dim(\mathcal{H}) \leq d$ for some constant d, then problem MinRED is solvable in polynomial time, while problem MaxIRR can be solved in incremental polynomial time.

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- For a hypergraph $\mathcal{H} \subseteq 2^V$ and subset $S \subseteq V$ define

$$\mathcal{H}_S \;=\; \{H \in \mathcal{H} \mid H \subseteq S\}$$

- Let us call $\mathcal{H}$ **cover-complete** (CC) if

$$\forall H \in \mathcal{H} \quad \text{we have} \quad U(\mathcal{H}_{V \setminus H}) = V \setminus H$$

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Problem MaxIRR can be solved in incremental quasi-polynomial time if the input is restricted to CC hypergraphs.

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Proofs: *MinRED* is co-NP-complete

- Given $\mathcal{H} \subseteq 2^V$, we define

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Lemma 6

If $\mathcal{R} \subseteq 2^V$ is a minimal redundant hypergraph, then $\text{core}(\mathcal{R}) \neq \emptyset$.

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- Define $N = \{1, 2, ..., n\}$, $Z = N \cup \Phi$, and

$$\begin{aligned} U &= \{(u', C') \mid u' \in L,\ C' \in \Phi,\ u' \in C'\} \\ X(u) &= \{u\} \cup \{i(u)\} \cup \{(u, C') \mid u \in C' \in \Phi\} \quad \forall u \in L \\ Y(C) &= \{C\} \cup \{(u', C) \mid u' \in C\} \quad \forall C \in \Phi \end{aligned}$$

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Lemma 7

- $|\{\mathcal{R} \in MinRED(\mathcal{H}_\Phi) \mid Z \notin core(\mathcal{R})\}| = 1 + 2|\Phi| + 3n$.
- *There exists $\mathcal{R} \in MinRED(\mathcal{H}_\Phi)$ with $Z \in core(\mathcal{R})$ iff Φ is consistent.*

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$$\begin{aligned} U &= \{(u', C') \mid u' \in L,\ C' \in \Phi,\ u' \in C'\} \\ X(u) &= \{u\} \cup \{i(u)\} \cup \{(u, C') \mid u \in C' \in \Phi\} \quad \forall u \in L \\ Y(C) &= \{C\} \cup \{(u', C) \mid u' \in C\} \quad \forall C \in \Phi \end{aligned}$$

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$$\mathcal{H}_\Phi = \{L\} \cup \{Z\} \cup \{X(u') \mid u' \in L\} \cup \{Y(C') \mid C' \in \Phi\}$$

]

Lemma 7

- $|\{\mathcal{R} \in MinRED(\mathcal{H}_\Phi) \mid Z \notin \text{core}(\mathcal{R})\}| = 1 + 2|\Phi| + 4n.$
- *There exists $\mathcal{R} \in MinRED(\mathcal{H}_\Phi)$ with $Z \in \text{core}(\mathcal{R})$ iff Φ is consistent.*

Proofs: $MinRED$ is co-NP-complete

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Proofs: $MaxIRR$ is NP-complete

- Consider a SAT instance $\Phi \subseteq 2^L$, $|C| \geq 2$ for all $C \in \Phi$, such that $\forall C \in \Phi\ \exists \{u, \overline{u}\} \subseteq L \setminus C$, for all $C, C' \in \Phi$ we have $C \cup C' \neq L$, and define

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P &= \{\{u, \overline{u}, v\} \mid u, v \in L,\ u \neq v \neq \overline{u}\} \\
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- Set $V = P \cup Q \cup R \cup S \cup T \cup U \subseteq 2^{L \cup \Phi}$, and for all $u \in L$ and $C \in \Phi$ define $X(u) = \{F \in V \mid F \ni u\}$ and $Y(C) = \{F \in V \mid F \ni C\}$, and set

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- $\forall x \in \{0,1\}^n$ *s.t.* $\Phi(x) = 1\ \exists \mathcal{I}_x \in MaxIRR(\mathcal{H}_\Phi)$.
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Proofs: For $\Delta(\mathcal{H}) = 2$ $MinRED$ is trivial, and $MaxIRR \Longleftrightarrow MinDOM$

- For all $H \in \mathcal{H}$ there exists at most one $\mathcal{R} \in MinRED(\mathcal{H})$ with $H \in \text{core}(\mathcal{R})$. Thus $|MinRED(\mathcal{H})| \leq |\mathcal{H}|$ and it is easy to generate it in polynomial time.
- If $\deg_{\mathcal{H}}(v) = 2$ for all $v \in V$, then define a graph $G_{\mathcal{H}} = (W, E)$ by setting $W = \mathcal{H}$, and $E = \{(H, H') \mid H, H' \in \mathcal{H},\ H \cap H' \neq \emptyset\}$.
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Proofs: For $\dim(\mathcal{H}) \leq d$ the family $MinRED(\mathcal{H})$ can be generated in polynomial time, while $MaxIRR(\mathcal{H})$ in incremental polynomial time

- If $\mathcal{R} \in MinRED(\mathcal{H})$, then $|\mathcal{R}| \leq d+1$, implying

$$|MinRED(\mathcal{H})| \leq \binom{|\mathcal{H}|}{d+1}$$

- Since $\dim(MinRED(\mathcal{H})) \leq d+1$, its dual $MaxIRR(\mathcal{H}) = MinTR(MinRED(\mathcal{H}))$ can be generated in incremental polynomial time (Eiter and Gottlob, 1995)

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Proofs: For CC hypergraphs $\mathcal{H}$ the set $MaxIRR(\mathcal{H})$ can be generated in incremental quasi-polynomial time

- To $\mathcal{R} \in MinRED(\mathcal{H})$ associate a unique $H_{\mathcal{R}} \in \text{core}(\mathcal{R})$.
- $\mathcal{R} \setminus \{H_{\mathcal{R}}\}$ is a minimal cover of $H_{\mathcal{R}}$.
- By property CC, the family $(\mathcal{R} \setminus \{H_{\mathcal{R}}\}) \cup \{H \in \mathcal{H} \mid H \cap H_{\mathcal{R}} = \emptyset\}$ is a cover of V. Thus, there exists a minimal cover $\mathcal{C} = \mathcal{C}_{\mathcal{R}} \in MinCOV(\mathcal{H})$ satisfying

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- To different minimal redundant sets $\mathcal{R}, \mathcal{R}' \in MinRED(\mathcal{H})$ with $H_{\mathcal{R}} = H_{\mathcal{R}'}$ we get different minimal covers $\mathcal{C}_{\mathcal{R}} \neq \mathcal{C}_{\mathcal{R}'}$, implying

$$|MinRED(\mathcal{H})| \leq |\mathcal{H}| \cdot |MinCOV(\mathcal{H})|$$

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- Then, by joint generation (Bioch and Ibaraki, 1995; Gurvich and Khachiyan, 1999), $MaxIRR(\mathcal{H})$ can be generated in total quasi-polynomial time.

Proofs: For CC hypergraphs $\mathcal{H}$ the set $MaxIRR(\mathcal{H})$ can be generated in incremental quasi-polynomial time

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Proofs: For CC hypergraphs $\mathcal{H}$ the set $MaxIRR(\mathcal{H})$ can be generated in incremental quasi-polynomial time

- Recall ...

$$|MinRED(\mathcal{H})| \ \leq \ |\mathcal{H}| \cdot |MinCOV(\mathcal{H})|$$

- For all subfamilies $MinCOV(\mathcal{H}) \subseteq \mathcal{G} \subseteq MaxIRR(\mathcal{H})$ we have

$$\begin{aligned} |MinTR(\mathcal{G}) \cap MinRED(\mathcal{H})| \ &\leq \ |MinRED(\mathcal{H})| \\ &\leq \ |\mathcal{H}| \cdot |MinCOV(\mathcal{H})| \ \leq \ |\mathcal{H}| \cdot |\mathcal{G}| \end{aligned}$$

- Consequently, $MaxIRR(\mathcal{H}) \setminus MinCOV(\mathcal{H})$ is **uniformly dual bounded**.
- Thus, we can start generating $MinCOV(\mathcal{H})$ in incremental quasi-polynomial time (monotone dualization, Fredman and Khachiyan, 1996) and then continue with joint generation by the above inequality in incremental quasi-polynomial time (Boros, Gurvich, Khachiyan, and Makino, 2000).

Proofs: For CC hypergraphs $\mathcal{H}$ the set $MaxIRR(\mathcal{H})$ can be generated in incremental quasi-polynomial time

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- Thus, we can start generating $MinCOV(\mathcal{H})$ in incremental quasi-polynomial time (monotone dualization, Fredman and Khachiyan, 1996) and then continue with joint generation by the above inequality in incremental quasi-polynomial time (Boros, Gurvich, Khachiyan, and Makino, 2000).

Proofs: For CC hypergraphs $\mathcal{H}$ the set $MaxIRR(\mathcal{H})$ can be generated in incremental quasi-polynomial time

- Recall ...

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- For all subfamilies $MinCOV(\mathcal{H}) \subseteq \mathcal{G} \subseteq MaxIRR(\mathcal{H})$ we have

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